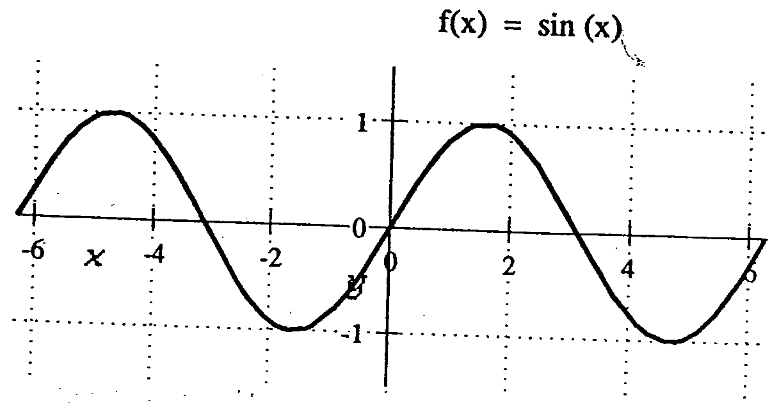
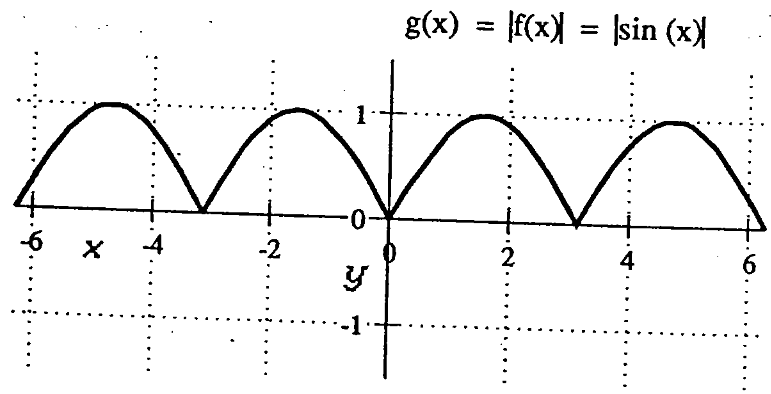
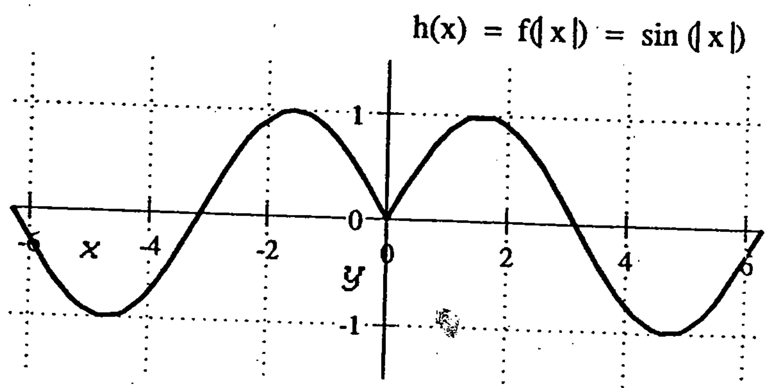
PROPERTIES OF FUNCTIONS







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GRAPHS OF ELEMENTARY FUNCTIONS

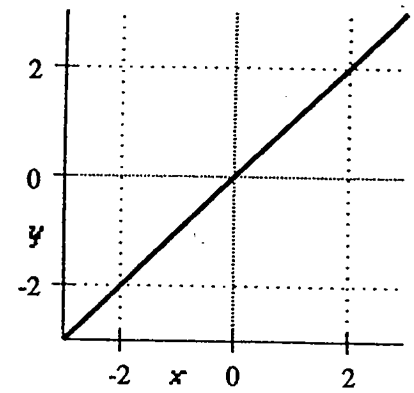
$f(x) = x$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

Intercepts: (0, 0)

Symmetry: origin



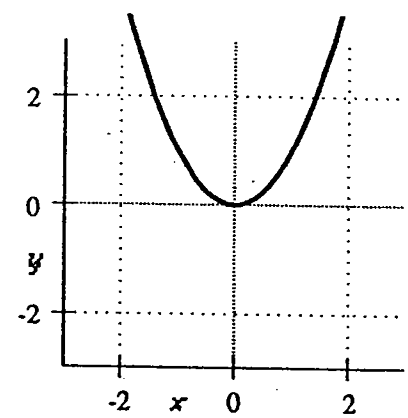
$f(x) = x^2$

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

Intercepts: (0,0)

Symmetry: y-axis



$f(x) = x^n$,

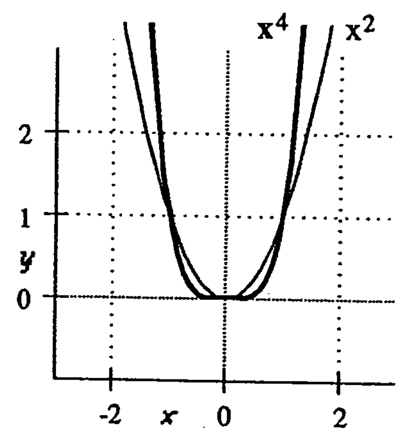
n an even positive integer

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

Intercepts: (0, 0)

Symmetry: y-axis



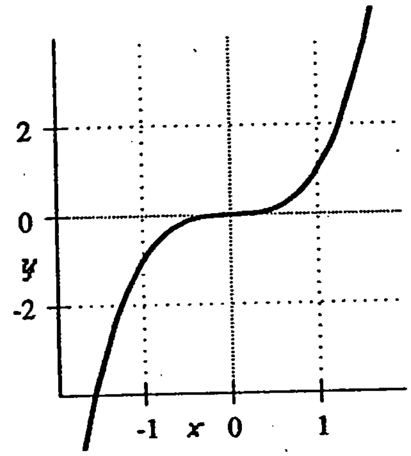
$f(x) = x^3$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

Intercepts: (0, 0)

Symmetry: origin



$f(x) = x^n$

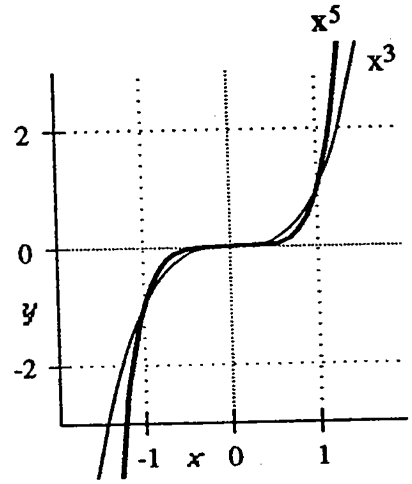
n a positive odd integer

Domain: $(-\infty,\infty)$

Range: $(-\infty,\infty)$

Intercepts: (0,0)

Symmetry: origin

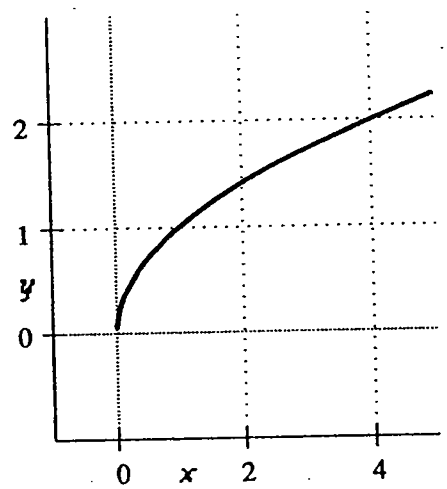


$f(x) = \sqrt{x}$

Domain: $[0, \infty)$

Range: $[0, \infty)$

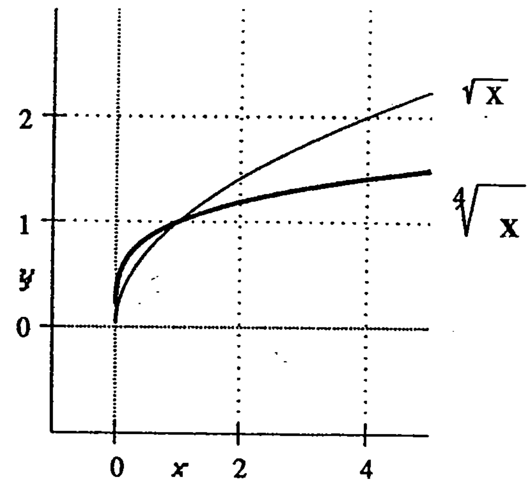
Intercepts: (0, 0)



$f(x) = \sqrt[n]{x}$

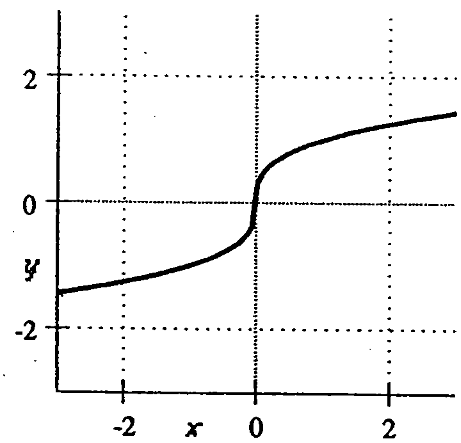
n an even positive integer

Domain: $[0, \infty)$
Range: $[0, \infty)$
Intercepts: $(0, 0)$



$f(x) = \sqrt[3]{x}$

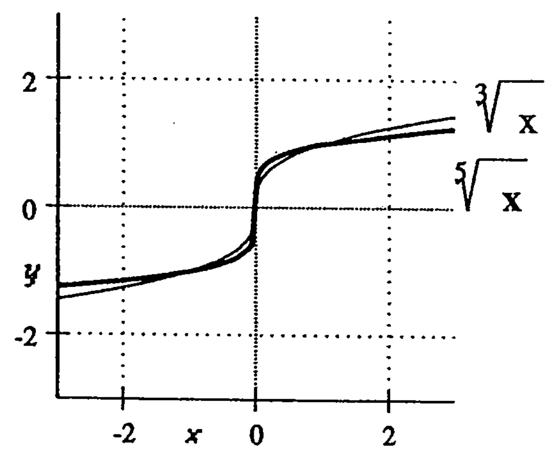
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$
Intercepts: $(0, 0)$
Symmetry: origin



$f(x) = \sqrt[n]{x}$

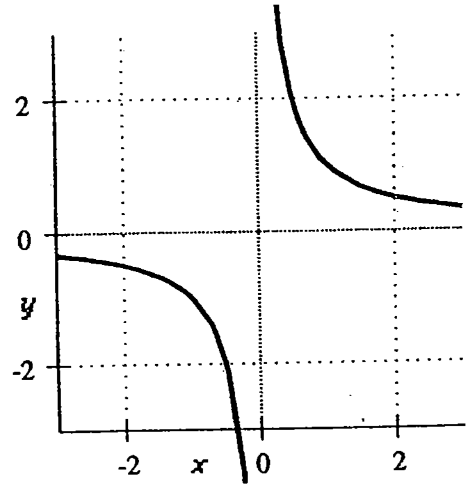
n a positive odd integer

Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$
Intercepts: $(0, 0)$
Symmetry: origin



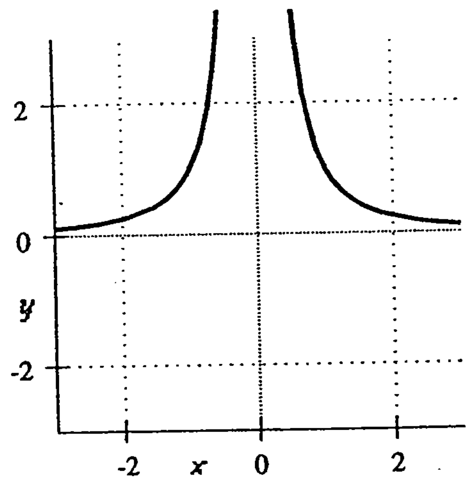
$$f(x) = \frac{1}{x}$$

Domain:	$(-\infty, 0) \cup (0, \infty)$
Range:	$(-\infty, 0) \cup (0, \infty)$
Intercepts:	none
Symmetry:	origin
Asymptotes:	$x = 0$, $y = 0$



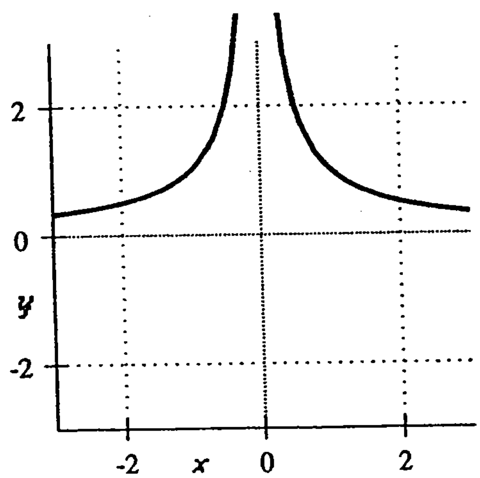
$$f(x) = \frac{1}{x^2}$$

Domain:	$(-\infty, 0) \cup (0, \infty)$
Range:	$(0, \infty)$
Intercepts:	none
Symmetry:	y-axis
Asymptotes:	$x = 0$, $y = 0$



$$f(x) = \frac{1}{|x|}$$

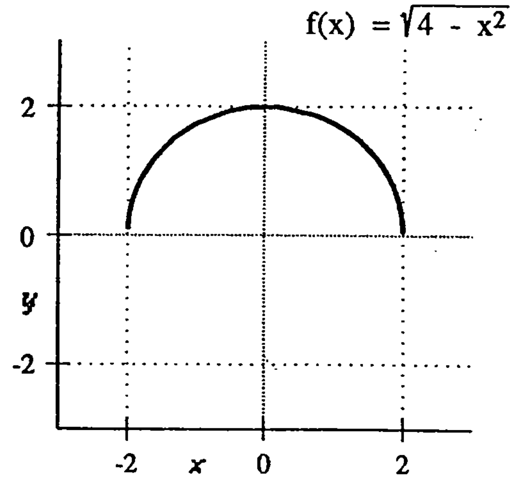
Domain:	$(-\infty, 0) \cup (0, \infty)$
Range:	$(0, \infty)$
Intercepts:	none
Symmetry:	y-axis
Asymptotes:	$x = 0$, $y = 0$



$f(x) = \sqrt{a^2 - x^2}$

Semi-circle of radius a

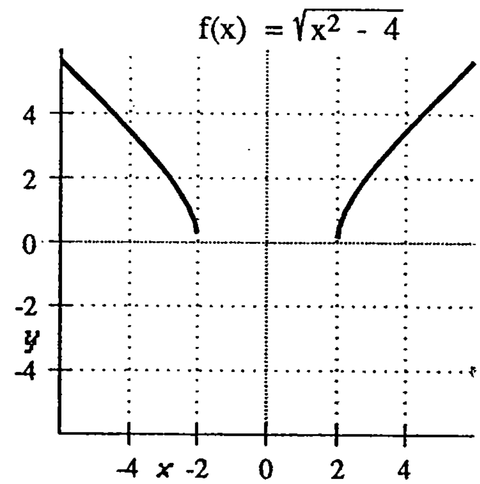
Domain: $[-a, a]$
Range: $[0, a]$
Intercepts: $(-a, 0), (a, 0), (0, a)$
Symmetry: y-axis



$f(x) = \sqrt{x^2 - a^2}$

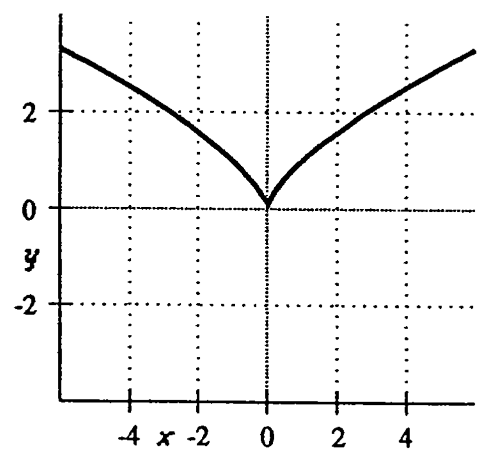
positive part of a hyperbola

Domain: $(-\infty, -a] \cup [a, \infty)$
Range: $[0, \infty)$
Intercepts: $(-a, 0), (a, 0)$
Symmetry: y-axis



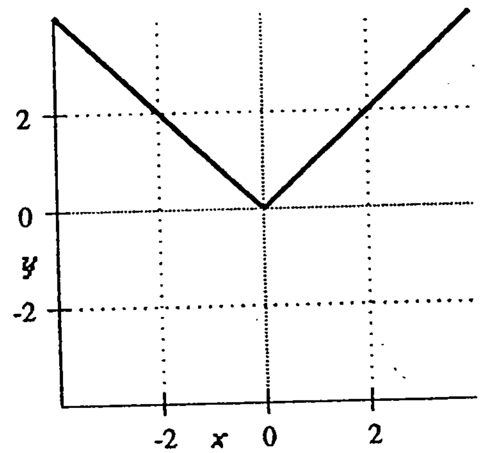
$f(x) = x^{\frac{2}{3}}$

Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
Intercepts: $(0, 0)$
Symmetry: y-axis



$f(x) = |x|$

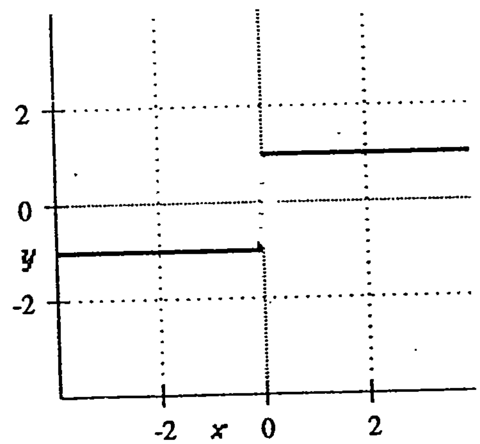
Domain:	$(-\infty, \infty)$
Range:	$[0, \infty)$
Intercepts:	(0, 0)
Symmetry:	y-axis



$f(x) = \frac{|x|}{x}$

Step function

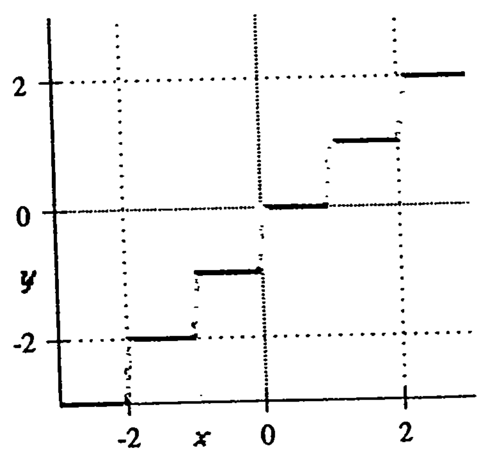
Domain:	$(-\infty, 0) \cup (0, \infty)$
Range:	$y = -1$, $y = 1$ only
Intercepts:	None
Symmetry:	origin



$f(x) = [\,x\,]$

Greatest integer $\leq x$

Domain:	$(-\infty, \infty)$
Range:	the set of all integers
Intercepts:	(0, 0)
Symmetry:	none



$f(x) = b^x \quad b > 1$

exponential function

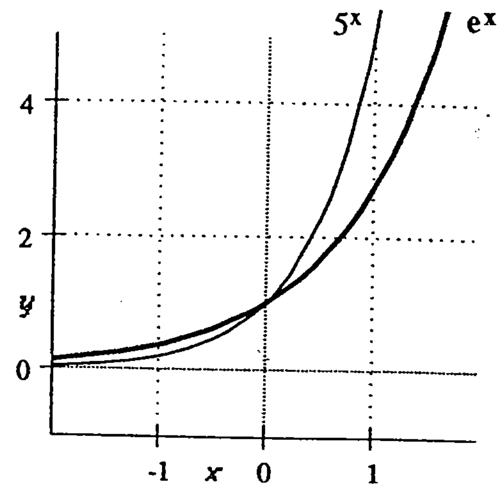
Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

Intercepts: (0, 1)

Symmetry: none

Asymptotes: $y = 0$



$f(x) = b^{-x}, \quad b > 1$

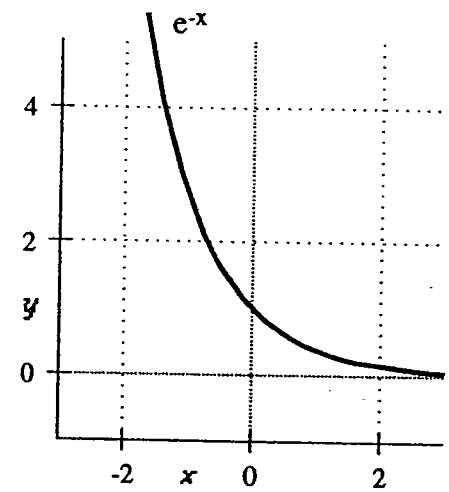
Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

Intercepts: (0, 1)

Symmetry: none

Asymptotes: $y = 0$



$f(x) = \log_b x, \quad b > 1$

logarithmic function

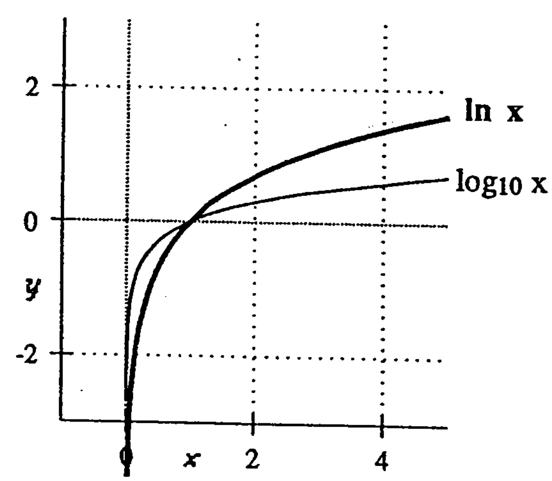
Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

Intercepts: (1, 0)

Symmetry: none

Asymptotes: $x = 0$



$f(x) = \sin x$

Domain: $(-\infty, \infty)$

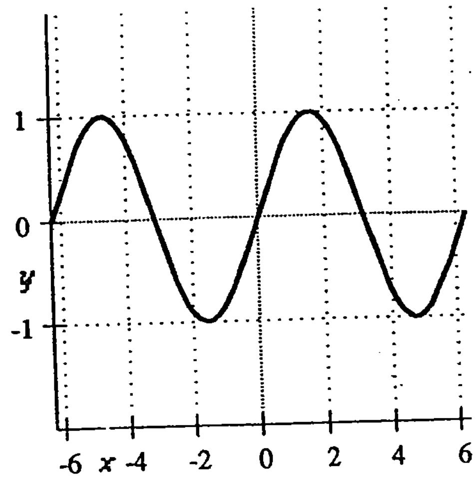
Range: $[-1, 1]$

Intercepts: $x = n\pi$, n an integer

$y = 0$

Symmetry: origin

Period: 2π



$f(x) = \cos(x)$

Domain: $(-\infty, \infty)$

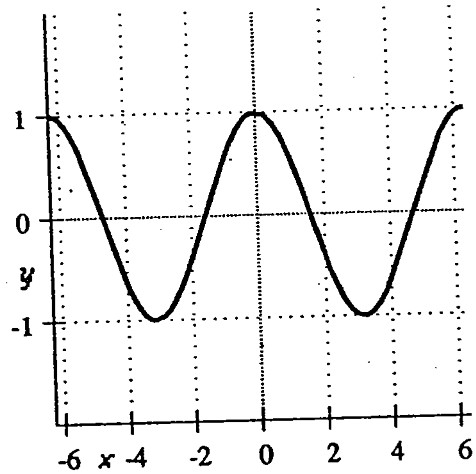
Range: $[-1, 1]$

Intercepts: $x = (2n+1)\frac{\pi}{2}$

$y = 1$

Symmetry: y-axis

Period: 2π



$f(x) = \tan x$

Domain: $x \neq (2n+1)\frac{\pi}{2}$

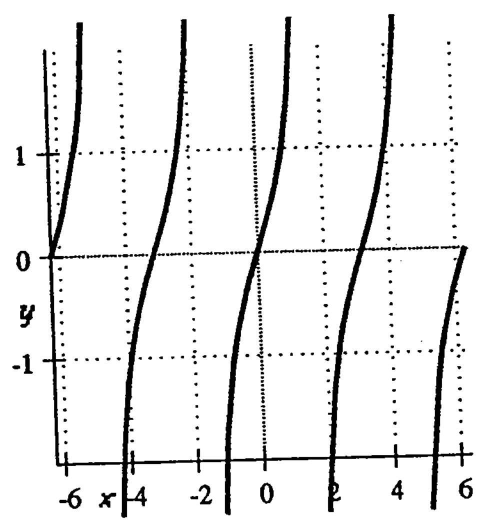
Range: $(-\infty, \infty)$

Intercepts: $x = n\pi$

$y = 0$

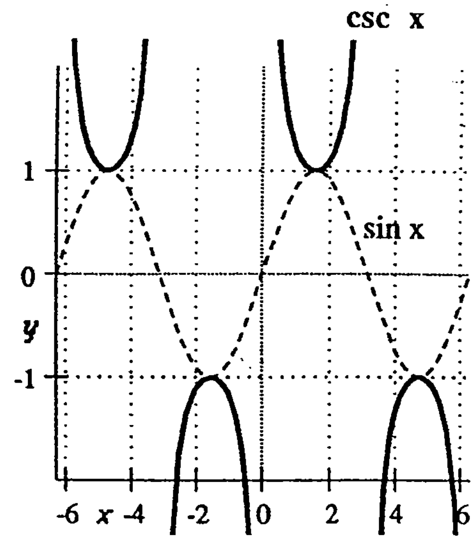
Symmetry: origin

Period: π



$f(x) = \csc x = \frac{1}{\sin x}$

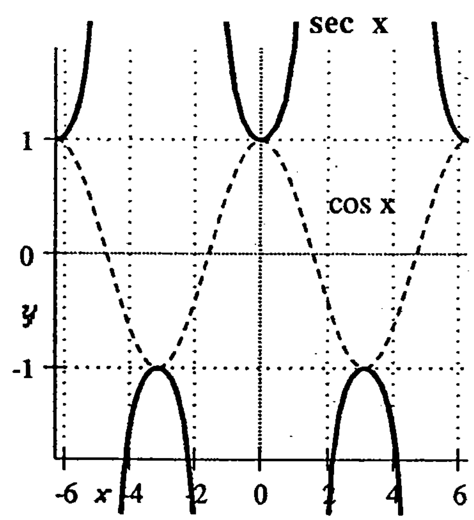
Domain:	$x \neq n\pi$
Range:	$\lvert y \rvert \geq 1$
Intercepts:	none
Symmetry:	origin
Period:	2π



$f(x) = \sec x = \frac{1}{\cos x}$

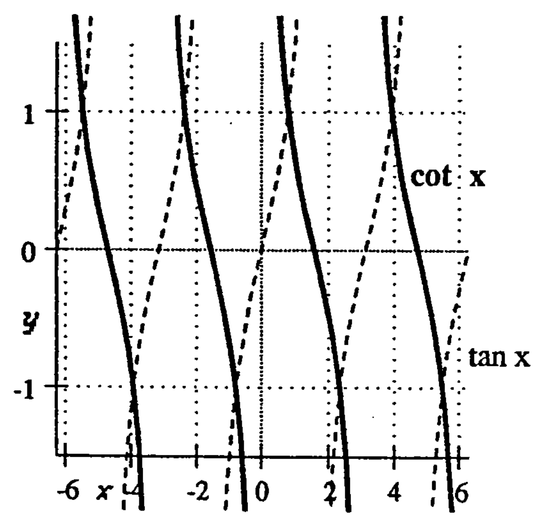
Domain: $x \neq (2n+1)\frac{\pi}{2}$

Range:	$\lvert y \rvert \geq 1$
Intercepts:	$y = 1$
Symmetry:	y-axis
Period:	2π



$f(x) = \cot x = \frac{1}{\tan x}$

Domain:	$x \neq n\pi$
Range:	$(-\infty, \infty)$
Intercepts:	$x = (2n+1)\frac{\pi}{2}$
Symmetry:	origin
Period:	π



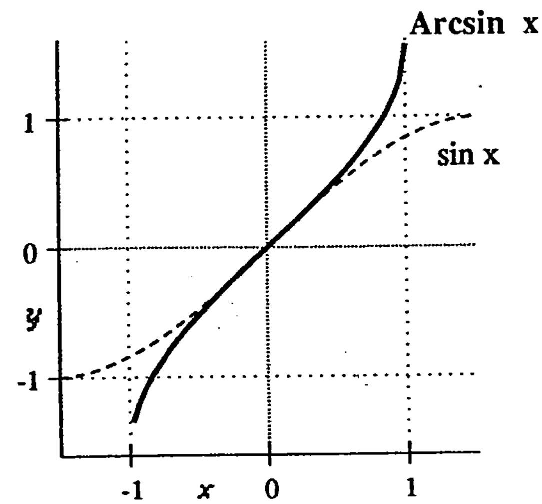
$f(x) = \text{Arcsin } x$

Domain: $[-1, 1]$

Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Intercepts: $(0, 0)$

Symmetry: origin



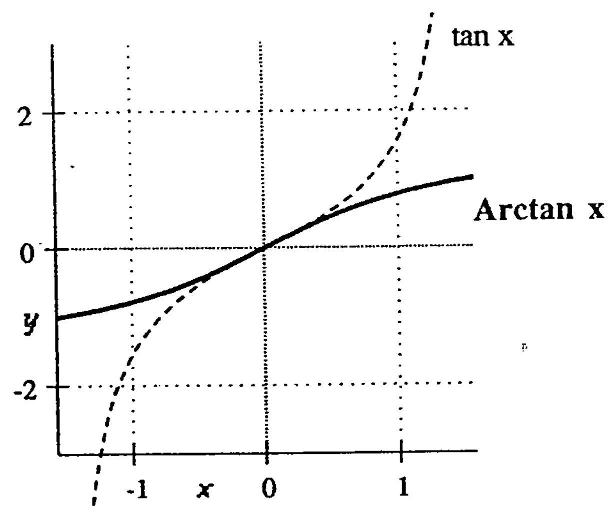
$f(x) = \text{Arctan } x$

Domain: $(-\infty, \infty)$

Range: $(-\frac{\pi}{2}, \frac{\pi}{2})$

Intercepts: $(0, 0)$

Symmetry: origin



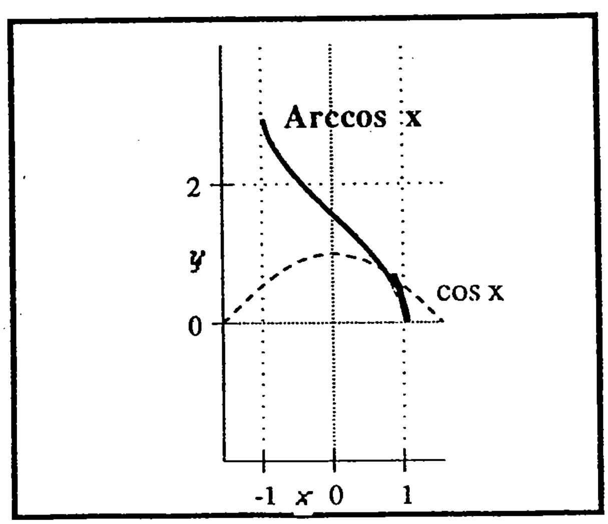
$f(x) = \text{Arccos } x$

Domain: $[-1, 1]$

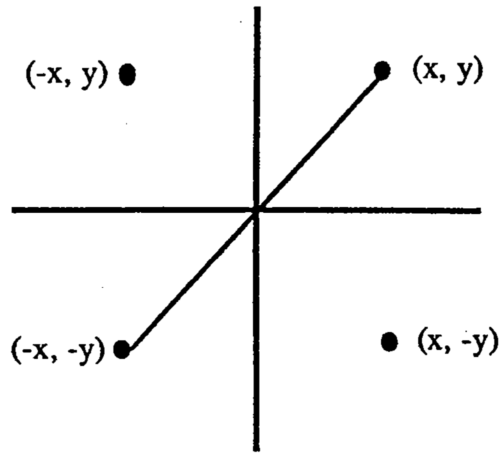
Range: $[0, \pi]$

Intercepts: $(0, \frac{\pi}{2})$

Symmetry: None



SYMMETRY



The diagram above shows the necessary conditions for symmetry about the axes and the origin.

The graph of a relation is symmetric to

1. **the y-axis if for all points (x, y) on its graph, the point (-x, y) is also on the graph.**
2. **the x-axis if for all points (x, y) on its graph, the point (x, -y) is also on the graph.**
3. **the origin if for all points (x, y) on its graph, the point (-x, -y) is also on the graph.**

For symmetry to the axes, the axis can be regarded as the surface of a mirror. For point symmetry to the origin, if we draw a line from the point (x, y) through the origin, the point (-x, -y) will lie on the line an equal distance on the other side of the origin.

RULES FOR DETERMINING SYMMETRY

The graph of a relation is symmetric to

1.	**the y-axis if replacing x by -x yields an equivalent relation.**
2.	**the x-axis if replacing y by -y yields an equivalent relation.**
3.	**the origin if replacing x by -x and y by -y yields an equivalent relation.**

EXAMPLE. Determine the symmetry of the following relations:

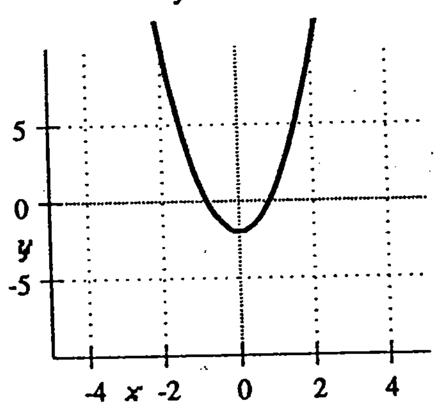
a. $y = 3x^2 - 2$
b. $y = 4x^3 - 1$
c. $y = x^3 - x$
d. $x^2 + y^2 = 1$

a. To y-axis YES $\quad y = 3(-x)^2 - 2 \quad \text{-->} \quad y = 3x^2 - 2$

To x-axis NO $\quad -y = 3x^2 - 2 \quad \text{-->} \quad y = -(3x^2 - 2)$

To origin NO $\quad -y = 3(-x)^2 - 2 \quad \text{-->} \quad -y = 3x^2 - 2$

$y = 3x^2 - 2$

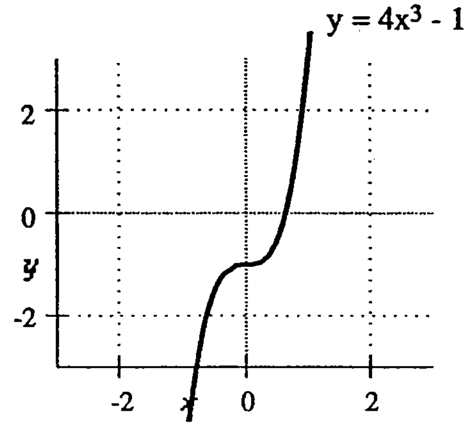


Symmetry to the y-axis

b. $y = 4x^3 - 1$

To y-axis: $y = 4(-x)^3 - 1$ --> $y = -4x^3 - 1$
No.
To x-axis: $-y = 4x^3 - 1$ --> $y = -4x^3 + 1$
No.
To origin: $-y = 4(-x)^3 - 1$ --> $y = 4x^3 + 1$
No.

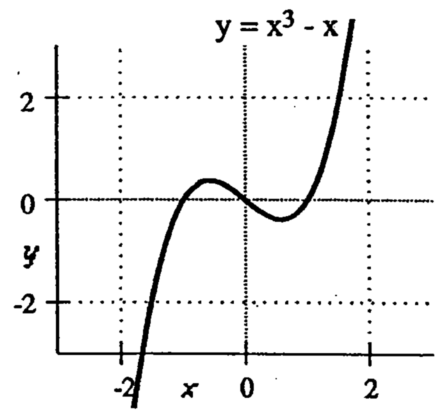
No symmetry.



c. $y = x^3 - x$

To y-axis: $y = (-x)^3 - (-x)$ --> $y = -x^3 + x$
No.
To x-axis $-y = x^3 - x$ --> $y = -x^3 + x$
No.
To origin: $-y = (-x)^3 - (-x)$ --> $y = x^3 - x$
Yes.

Symmetry to origin

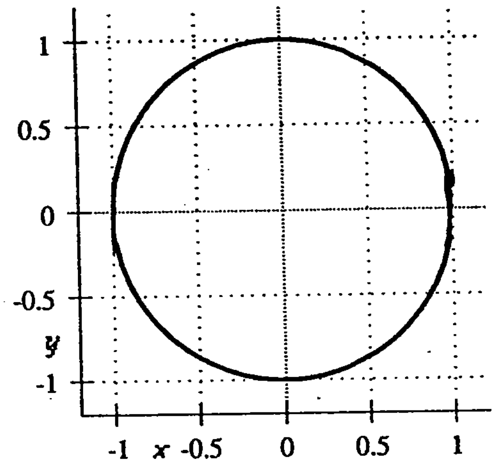


d. $x^2 + y^2 = 1$

To y-axis: $(-x)^2 + y^2 = 1$ --> $x^2 + y^2 = 1$
Yes.
To x-axis: $x^2 + (-y)^2 = 1$ --> $x^2 + y^2 = 1$
Yes.
To origin: $(-x)^2 + (-y)^2 = 1$ --> $x^2 + y^2 = 1$

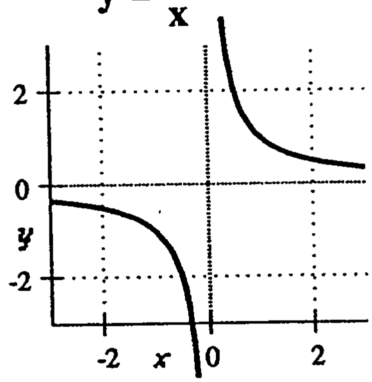
Yes.

Symmetric to both axes and origin.



Note that if the graph of a relation is symmetric to both the x and y-axes, it is symmetric to the origin. The converse is not true. A relation may be symmetric to the origin, but not to the axes. Consider the graph of

$$y = \frac{1}{x}$$



The graph is symmetric to the origin, but not to either axes.

ODD AND EVEN FUNCTIONS

DEFINITION

1. **A function is said to be even if, for all x in its domain, $f(-x) = f(x)$.**

2. **A functions is said to be odd if, for all x in its domain, $f(-x) = -f(x)$.**

Symmetry of Odd and Even functions

From our previous discussion of symmetry, $f(-x) = f(x)$ implies that if we replace the x-coordinate by $-x$, the y-coordinate remains the same. This is the equivalent of saying that if (x, y) is on the graph of f, $(-x, y)$ is also on the graph. Therefore, *the graph of an even function is symmetric to the y-axis.*

Similarly, $f(-x) = -f(x)$ implies that if we replace the x-coordinate by $-x$, the y-coordinate becomes $-y$. This is the equivalent of saying that if (x, y) is on the graph of f, $(-x, -y)$ is also on the graph. Therefore, *the graph of an odd function is symmetric to the origin.*

Note that the graph of f cannot be symmetric to the x-axis, because then it would not be a function (It would fail the vertical line test for a function; for values of x, there would be two values of y).

Summarizing

1. **If f is even, that is, if $f(-x) = f(x)$, then the graph of f is symmetric to the y-axis.**

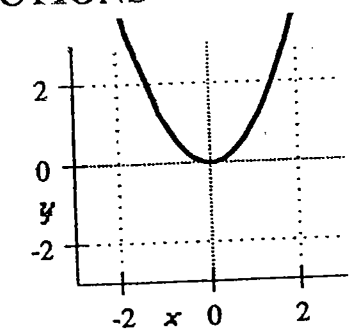
2. **If f is odd, that is, if $f(-x) = -f(x)$, then the graph of f is symmetric to the origin.**

3. **It is possible for a function to be neither even nor odd.**

EXAMPLES OF EVEN FUNCTIONS

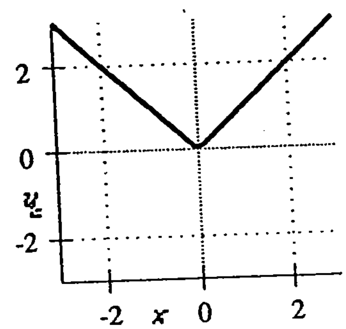
1. $f(x) = x^2$

   $f(-x) = (-x)^2 = x^2 = f(x)$



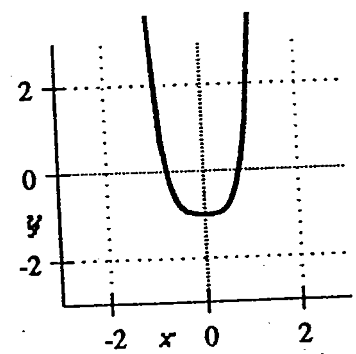
2. $f(x) = |x|$

   $f(-x) = |-x| = |x| = f(x)$



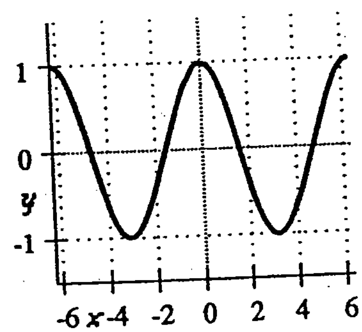
3. $f(x) = 3x^4 - 1$

   $f(-x) = 3(-x)^4 - 1 = 3x^4 - 1 = f(x)$



4. $f(x) = \cos(x)$

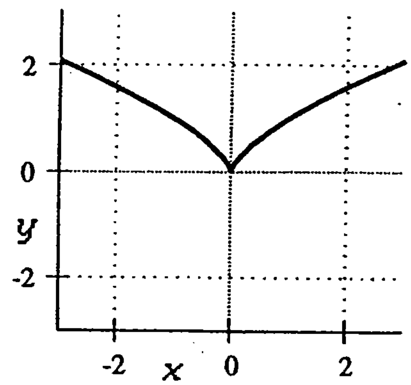
   $f(-x) = \cos(-x) = \cos(x) = f(x)$



5.

$f(x) = x^{2/3} = \sqrt[3]{x^2}$

$f(-x) = \sqrt[3]{(-x)^2} = \sqrt[3]{x^2} = f(x)$

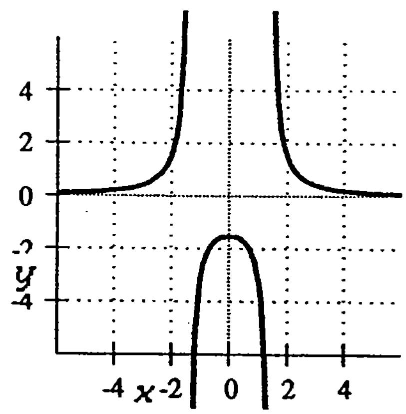


6.

$$f(x) = \frac{3x}{x^3 - 2x}$$

$$f(-x) = \frac{3(-x)}{(-x)^3 - 2(-x)} = \frac{-3x}{-x^3 + 2x}$$

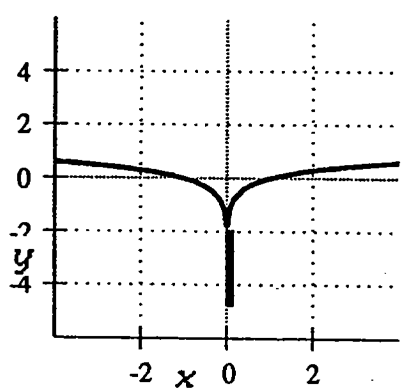
$$= \frac{-3x}{-(x^3 - 2x)} = \frac{3x}{x^3 - 2x} = f(x)$$



7.

$f(x) = \log|x|$

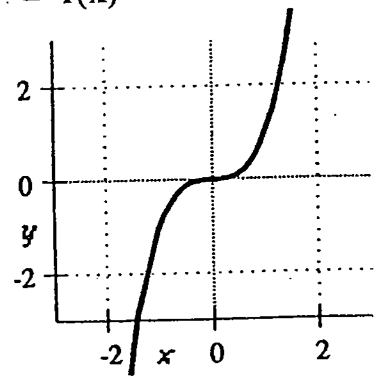
$f(-x) = \log|-x| = \log|x| = f(x)$



EXAMPLES OF ODD FUNCTIONS $f(-x) = -f(x)$

1. $f(x) = x^3$

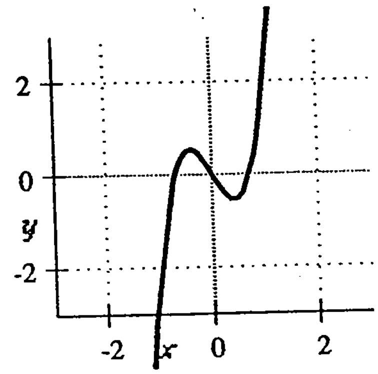
   $f(-x) = (-x)^3 = -x^3 = -f(x)$



2. $f(x) = 4x^3 - 2x$

   $f(-x) = 4(-x)^3 - 2(-x) = -4x^3 + 2x$

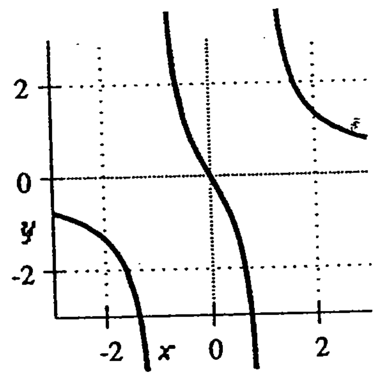
   $= -(4x^3 - 2x) = -f(x)$



3. $$f(x) = \frac{2x}{x^2 - 1}$$

   $$f(-x) = \frac{2(-x)}{(-x)^2 - 1} = \frac{-2x}{x^2 - 1}$$

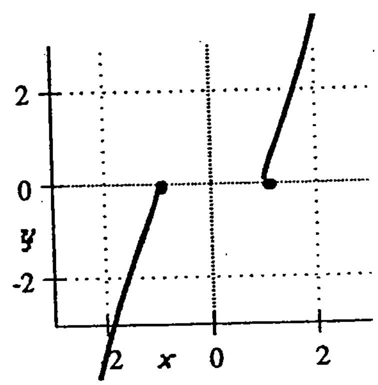
   $$= -\frac{2x}{x^2 - 1} = -f(x)$$



4. $$f(x) = x\sqrt{x^4 - 1}$$

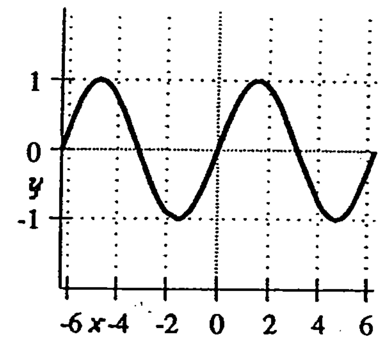
   $$f(-x) = (-x)\sqrt{(-x)^4 - 1}$$

   $$= -x\sqrt{x^4 - 1} = -f(x)$$



5. $f(x) = \sin(x)$

$f(-x) = \sin(-x) = -\sin(x) = -f(x)$

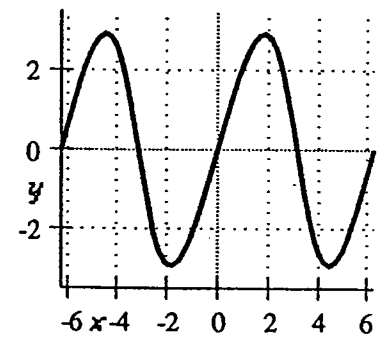


6.

$$f(x) = \frac{4\sin(x)}{\sqrt{2+\cos(x)}}$$

$$f(-x) = \frac{4\sin(-x)}{\sqrt{2+\cos(-x)}} = \frac{-4\sin(x)}{\sqrt{2+\cos(x)}}$$

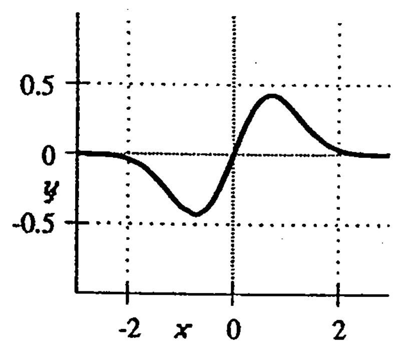
$$= -\frac{4\sin(x)}{\sqrt{2+\cos(x)}} = -f(x)$$



7.

$$f(x) = x\,e^{-x^2}$$

$$f(-x) = (-x)\,e^{-(-x)^2} = -x\,e^{-x^2} = -f(x)$$



POLYNOMIAL FUNCTIONS

If $f(x)$ is a polynomial function, i.e.,

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

where n is an integer ≥ 0, then

1. **f is an even function if all the exponents of the x-terms are even.**
2. **f is an odd function if all the exponents of the x-terms are odd.**
3. **f is neither even nor odd if the exponents of the x-terms are a mixture of even and odd integers.**

EXAMPLE: Determine whether each of the following polynomial functions are even, odd, or neither.

a. $f(x) = 3x^4 - 2x^2 + 7$

Note that the constant term 7 is equivalent to $7x^0$.

The exponents of x are all even. Therefore f is even.

$f(-x) = f(x)$

b. $f(x) = 3x^5 - 4x^3 - x$

The exponents of x are all odd. Therefore f is an odd function.

$f(-x) = -f(x)$

c. $f(x) = x^3 + 3x - 5$

The constant term -5 is equivalent to $-5x^0$.

The exponents of x are a mixture of odd and even integers.

Therefore f is neither even nor odd

$f(-x) = (-x)^3 + 3(-x) - 5 = -x^3 - 3x - 5 \neq f(x)$ or $-f(x)$

PRODUCT AND QUOTIENTS OF EVEN AND ODD FUNCTIONS

The rules for the product and quotients of even and odd functions are very similar to the rules for multiplication and division of signed numbers, if we think of an odd function as a negative and an even function as a positive.

> **1. The product or quotient of two even functions is even.**
> **2. The product or quotient of two odd function is even**
> **3. The product or quotient of an even and odd function is odd.**

We prove the first part of Rule 2.

Let $f(x)$ and $g(x)$ be odd functions and let $w(x) = f(x)g(x)$.
Prove that $w(x)$ is an even function.

$f(x)$ is odd implies $f(-x) = -f(x)$. $g(x)$ is odd implies $g(-x) = -g(x)$

$w(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = w(x)$. Therefore w is even.

EXAMPLES: Determine whether each of the following functions is even, odd, or neither.

a. $f(x) = x\sqrt{x^2+1}$

b. $f(x) = \dfrac{2x}{x^3 + 4x}$

c. $f(x) = x^3 \sin x$

d. $f(x) = \dfrac{\cos x}{x + 1}$

a. odd times even: f is odd.

b. odd divided by odd: f is even

c. odd times odd: f is even

d. even (cos x) divided by a function neither even nor odd.
f is neither even nor odd.

SYMMETRY OF THE ELEMENTARY TRIGONOMETRIC FUNCTIONS

Of the six elementary trigonometric functions, 2 are even, the cosine and its reciprocal, the secant. The other 4 are odd. This implies that

$\cos(-x) = \cos(x)$	symmetry to y-axis
$\sec(-x) = \sec(x)$	"
$\sin(-x) = -\sin(x)$	symmetry to origin
$\csc(-x) = -\csc(x)$	"
$\tan(-x) = -\tan(x)$	"
$\cot(-x) = -\cot(x)$	"

Remembering which functions are even and which are odd is a handy way of remembering the trigonometric identities above.

THE FUNCTION $f(|x|)$

The function $f(|x|)$, formed by everywhere replacing x in $f(x)$ by $|x|$, is *always* an even function, regardless of whether f is even, odd, or neither. This is because $f(|-x|) = f(|x|)$ for all x.

EXAMPLES:

$f(x) = \sin(x)$ is odd, but $g(x) = f(|x|) = \sin|x|$ is even.

$f(x) = x^4 + 3x^2 + 5$ is even, and so is $g(x) = |x|^4 + 3|x|^2 + 5$.

$f(x) = x + 1$ is neither even nor odd, but $g(x) = f(|x|) = |x| + 1$ is even.

The concept of even and odd functions will be important later on in calculus. Among other things, calculations involving the definite integral can be simplified if the function is even or odd.

TRANSFORMATIONS

Knowing the graphs of the elementary functions, we can quickly sketch the graphs of many other functions by recognizing them as transformations of the plane, either translations or reflections.

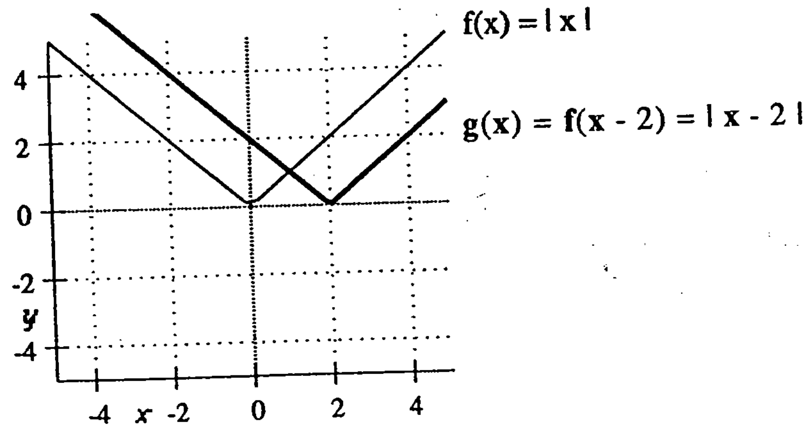
Horizontal Translations
The Graph of $f(x - h)$

The graph of $f(x - h)$, h a constant, is a horizontal translation of the graph of $f(x)$ by h units, to the right if $h > 0$ (h positive), to the left if $h < 0$ (h negative).

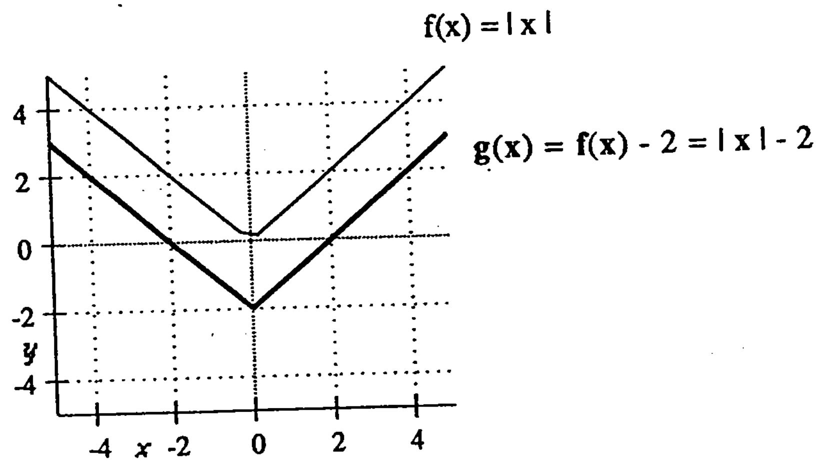
Vertical Translations
The graph of $f(x) + k$

The graph of $f(x) + k$, k a constant, is a vertical translation of the graph of $f(x)$ by k units, up if $k > 0$, down if $k < 0$.

g(x) = f(x - 2)   Horizontal translation 2 units to the right.

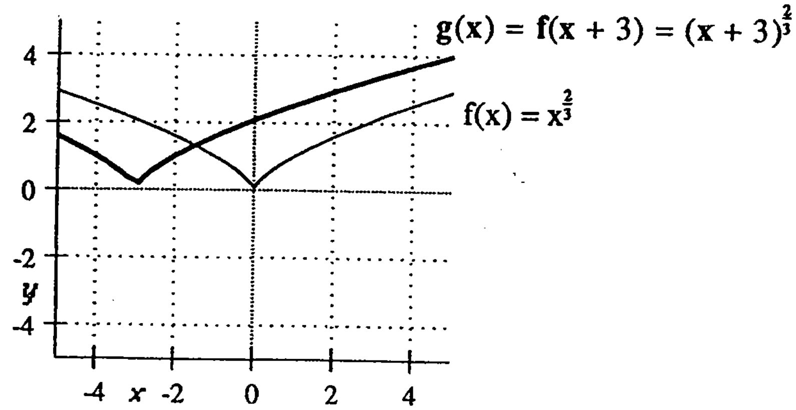


g(x) = f(x) - 2.   Vertical translation 2 units down.

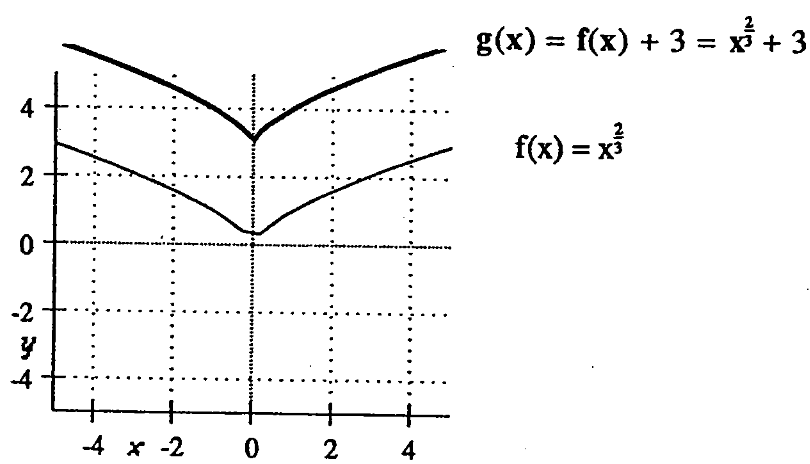


Note the difference between a horizontal and a vertical translation. For a horizontal translation, we replace x by x - h, whereas for a vertical translation we simply add a constant k to the original function.

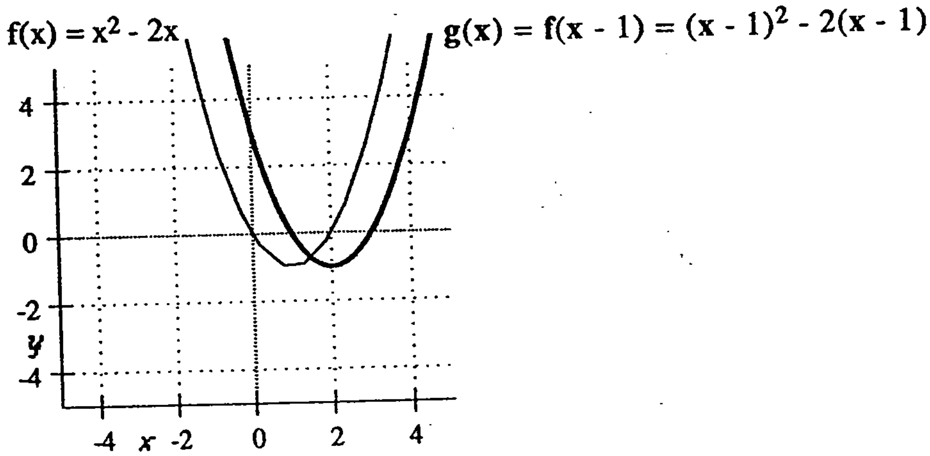
$g(x) = f(x + 3) = f(x - -3)$. *h is negative.* Horizontal translation 3 units to the left.



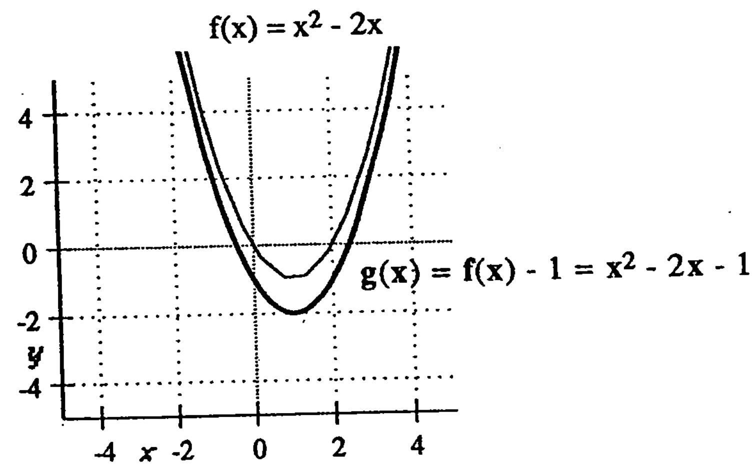
$g(x) = f(x) + 3$. Vertical translation 3 units up.



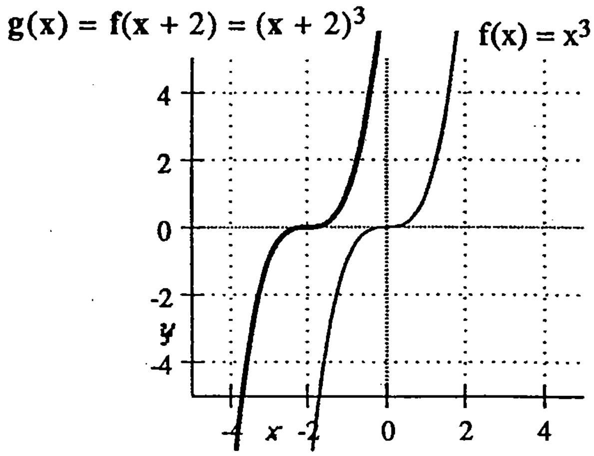
$g(x) = f(x - 1)$. Horizontal translation 1 unit to the right.



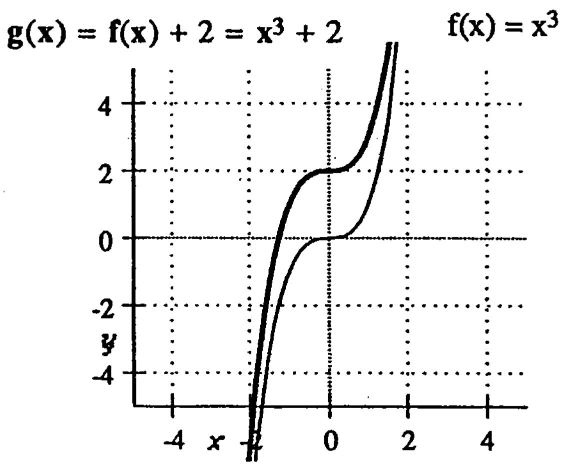
$g(x) = f(x) - 1$. Vertical translation 1 unit down.



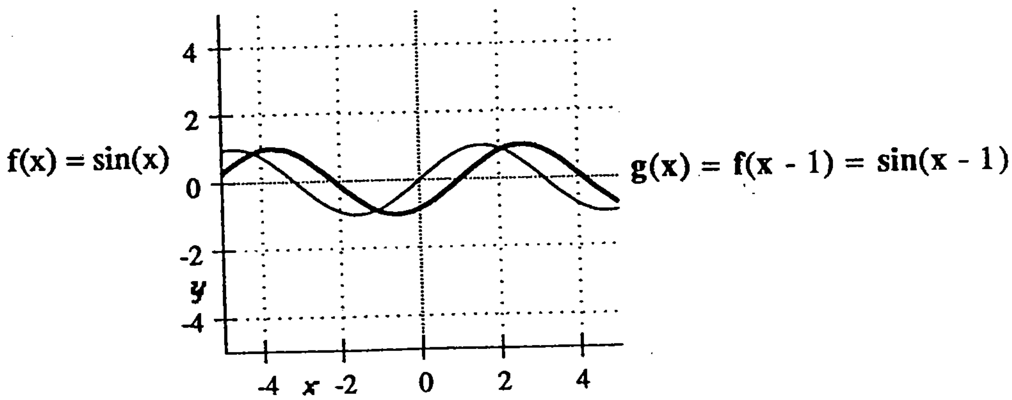
g(x) = f(x + 2) = f(x - -2)   Horizontal translation 2 units to the left.



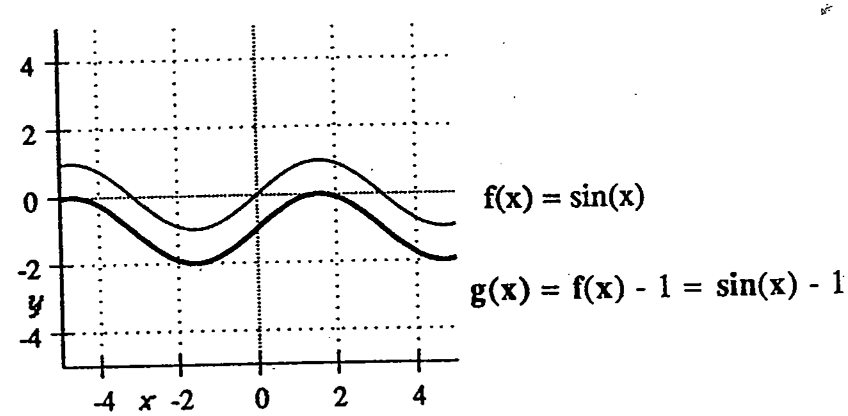
g(x) = f(x) + 2.   Vertical translation 2 units up.



$g(x) = f(x - 1)$ Horizontal translation 1 unit to the right.

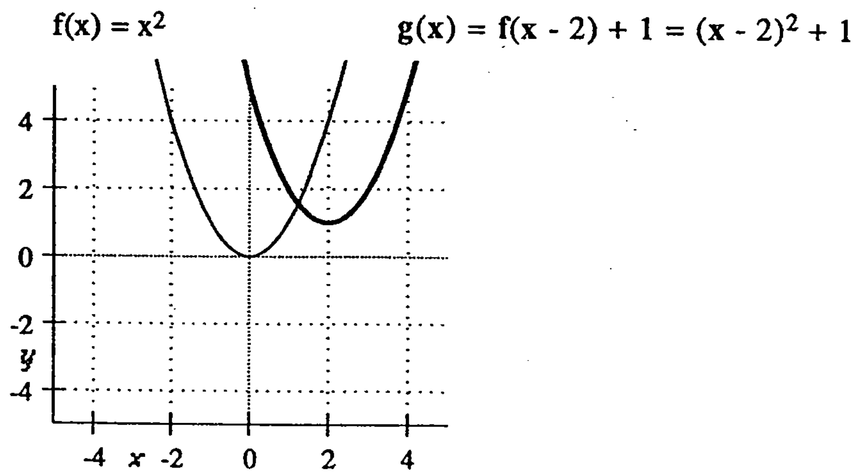


$g(x) = f(x) - 1$ Vertical translation 1 unit down.

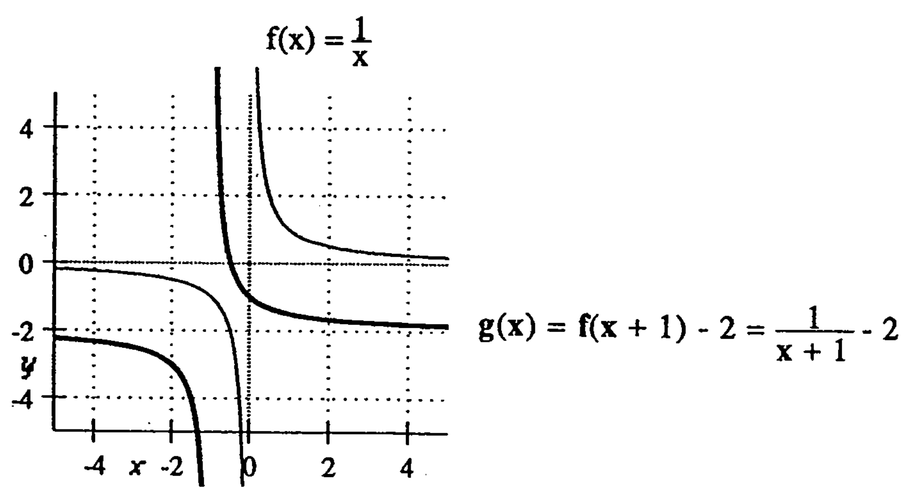


We can combine both horizontal and vertical translations.

$g(x) = f(x - 2) + 1$ Horizontal translation 2 units to the right.
Vertical translation 1 unit up.



$g(x) = f(x + 1) - 2$ Horizontal translation 1 unit to the left.
Vertical translation 2 units down.



REFLECTIONS ACROSS THE AXES

Reflections Across the Y-axis

The graph of f(-x) is a reflection of the graph of f(x) across the y-axis (that is, the graph of f(-x) is a mirror image of the graph of f(x) across the y-axis).

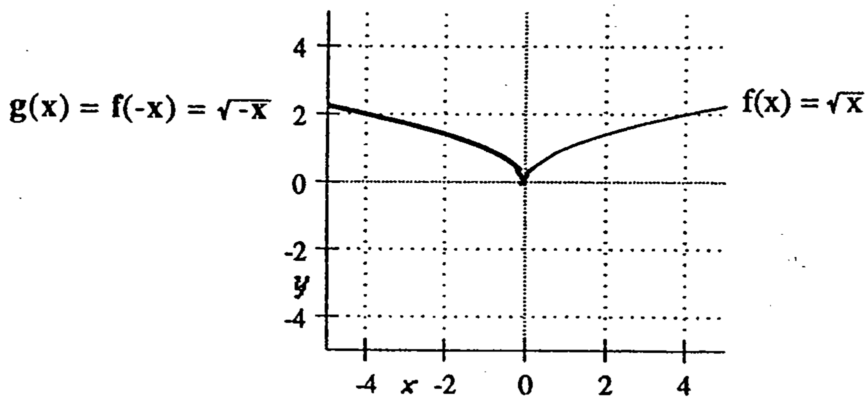
Note: Do not confuse this with the concept of an even function. f(x) is even and symmetric to the y-axis if, for all x, f(-x) = f(x). But in general, the graph of f(-x) is not necessarily symmetric to the y-axis (if f(x) is not.) The graph of f(-x) is the mirror image of f(x) across the y-axis. Of course, if f(x) is even to begin with, then the graph of f(-x) is a mirror image of itself. Sort of like a narcissistic function. The problem with being narcissistic is that it's so easy to fall in love with one-self. But I digress... What? You don't know who Narcissus was?

Reflections Across the X-axis.

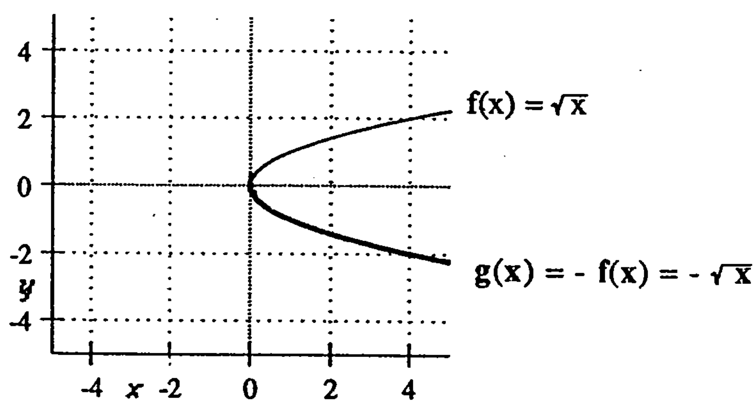
The graph of - f(x) is a reflection of the graph of f(x) across the x-axis (that is, the graph of - f(x) is the mirror image of the graph of f(x) across the x-axis.)

The above should make sense. f(x) is the y-coordinate. So making the y-coordinate the opposite while keeping x the same should give us a mirror image of the point on the other side of the x-axis.

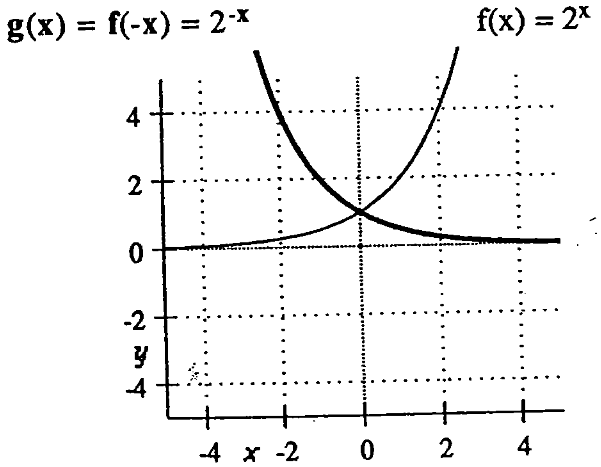
$g(x) = f(-x)$ Reflection across the y-axis.



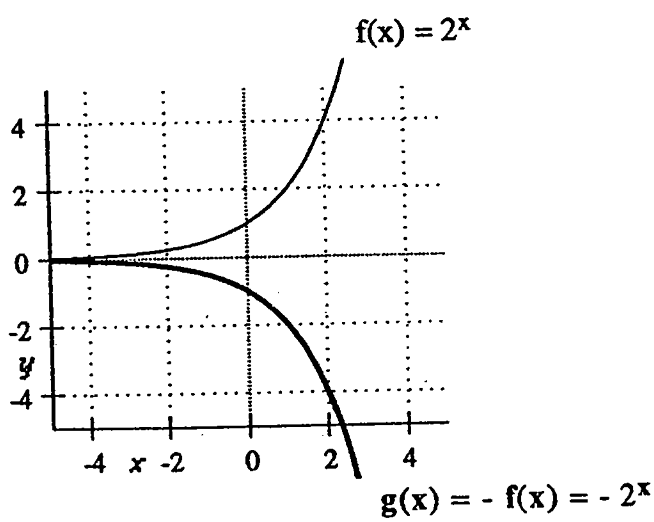
$g(x) = - f(x)$ Reflection across the x-axis.



$g(x) = f(-x)$ Reflection across the y-axis



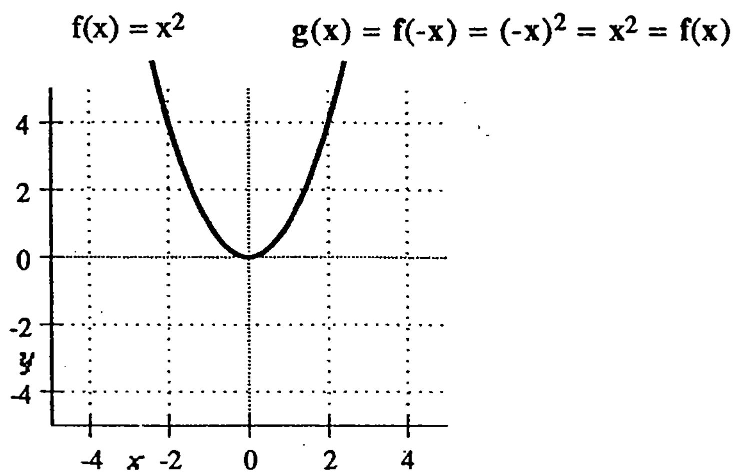
$g(x) = - f(x)$ Reflection across the x-axis.



If f(x) is an even function, then f(-x) = f(x) for all x, and the graph of f(-x) is identical to the graph of f(x).

$f(-x) = f(x)$ f even

$g(x) = f(-x) = f(x)$ The graphs of f and g are identical.

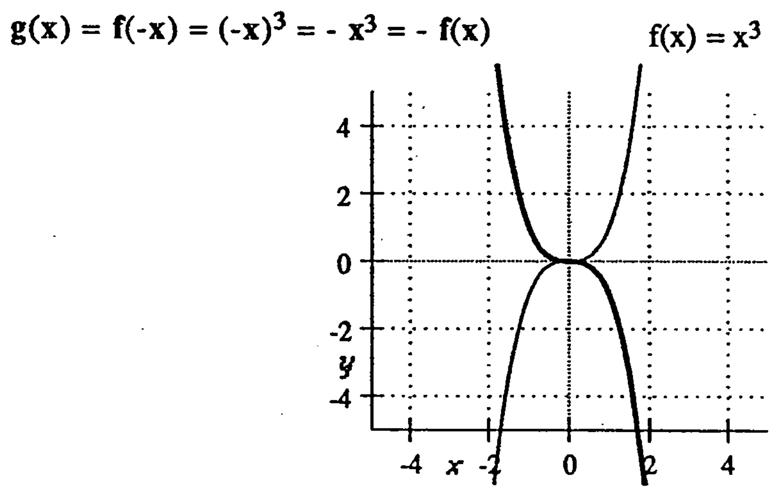


If f(x) is an odd function, then f(-x) = - f(x) for all x, and the graph of f(-x) is a reflection of the graph of f(x) across both the x and y-axes.

$f(-x) = - f(x)$ f odd

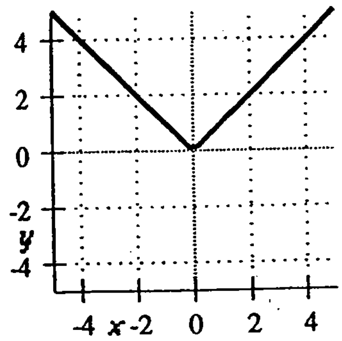
$g(x) = f(-x) = - f(x)$

Reflection across both x and y-axes.

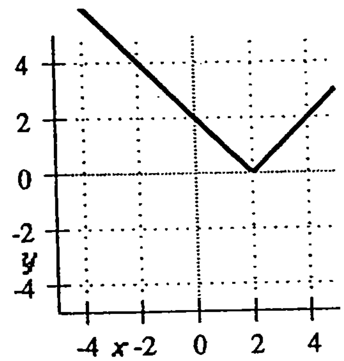


We can combine both translations and reflections.

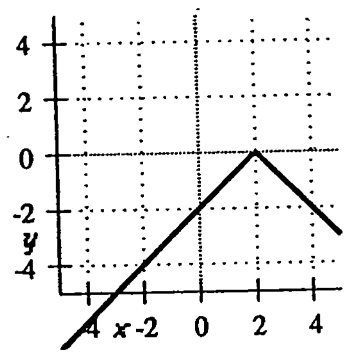
$f(x) = |x|$



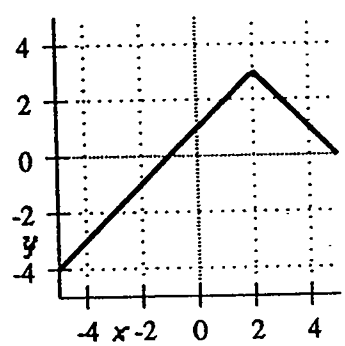
$f(x - 2) = |x - 2|$



$-f(x - 2) = -|x - 2|$



$-f(x - 2) + 3 = -|x - 2| + 3$

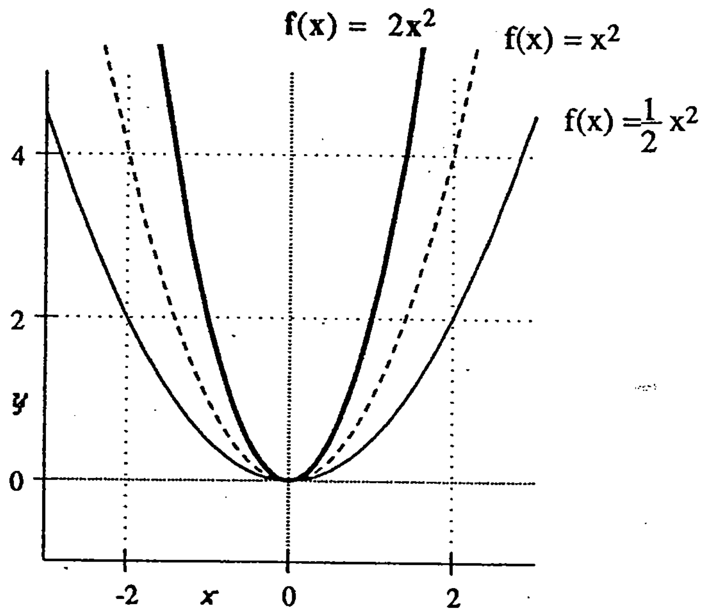


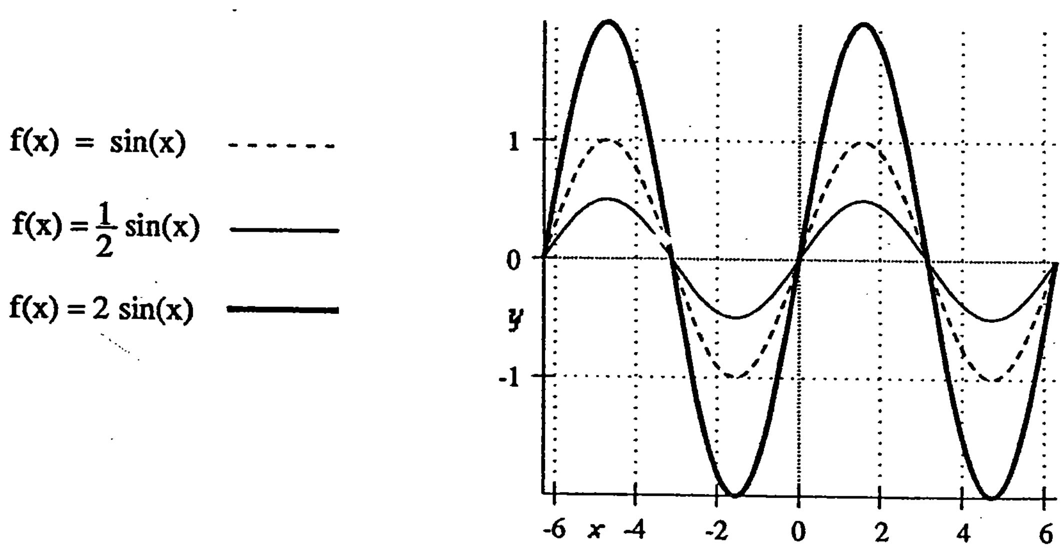
OTHER TRANSFORMATIONS

Vertical Stretching and Shrinking

The graph of $k\,f(x)$, k a constant, is a vertical stretching or shrinking of the graph of $f(x)$: A stretching, if $k > 1$; a shrinking if $k < 1$.

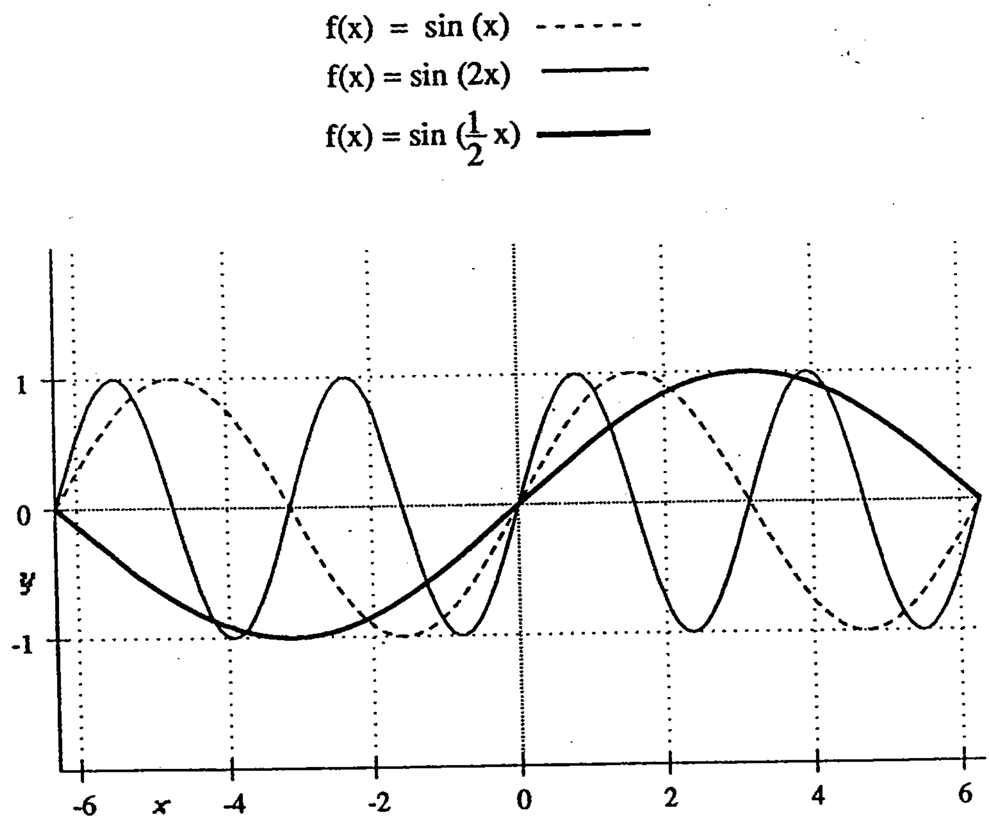
Basically, this means that for $k > 1$, the y-coordinate will increase more rapidly, and the graph will look steeper. For $k < 1$, the y-coordinate will increase less rapidly, and the graph will be less steep.





Horizontal Stretching and Shrinking

The graph of f(kx) is a horizontal stretching or shrinking of the graph of f(x); a stretching if $k < 1$, a shrinking if $k > 1$.



ABSOLUTE VALUE FUNCTIONS

Definition of absolute value:

$$|u| = \begin{cases} u, & \text{for } u \geq 0 \\ -u, & \text{for } u < 0 \end{cases}$$

EXAMPLE 1.

$$|x - 3| = \begin{cases} x - 3, & \text{for } x - 3 \geq 0 \\ -(x - 3), & \text{for } x - 3 < 0 \end{cases} = \begin{cases} x - 3, & \text{for } x \geq 3 \\ 3 - x, & \text{for } x < 3 \end{cases}$$

EXAMPLE 2.

$$|x^2 - 4| = \begin{cases} x^2 - 4, & \text{for } x^2 - 4 \geq 0 \\ -(x^2 - 4), & \text{for } x^2 - 4 < 0 \end{cases} = \begin{cases} x^2 - 4, & \text{for } x < -2 \text{ or } x > 2 \\ 4 - x^2, & \text{for } -2 < x < 2 \end{cases}$$

EXAMPLE 3.

$$|x|^2 - 5|x| = \begin{cases} x^2 - 5x, & \text{for } x \geq 0 \\ (-x)^2 - 5(-x), & \text{for } x < 0 \end{cases} = \begin{cases} x^2 - 5x, & \text{for } x \geq 0 \\ x^2 + 5x, & \text{for } x < 0 \end{cases}$$

EXAMPLE 4. Let the domain of $\sin x$ be $[0, 2\pi)$. Then

$$|\sin x| = \begin{cases} \sin x, & \text{for } \sin x \geq 0 \\ -\sin x, & \text{for } \sin x < 0 \end{cases} = \begin{cases} \sin x, & \text{for } 0 \leq x \leq \pi \\ -\sin x, & \text{for } \pi < x < 2\pi \end{cases}$$

There will be occasions in calculus when we will need the above techniques to rewrite absolute value functions without using absolute value notations. Often, doing so will simplify the problem.

THE GRAPH OF $|f(x)|$

The definition of absolute value gives us an insight into how to quickly sketch the graph of $|f(x)|$, if we know the graph of $f(x)$.

$$|f(x)| = \begin{cases} f(x), & \text{for } f(x) \geq 0 \\ -f(x), & \text{for } f(x) < 0 \end{cases}$$

For those values of x for which the y-coordinate is positive or 0, the graph of $|f(x)|$ is exactly the same as the graph of $f(x)$.

For those values of x for which the y-coordinate is negative, the graph of $|f(x)|$ is a reflection of the graph of $f(x)$ across the x-axis $[\,-f(x)\,]$.

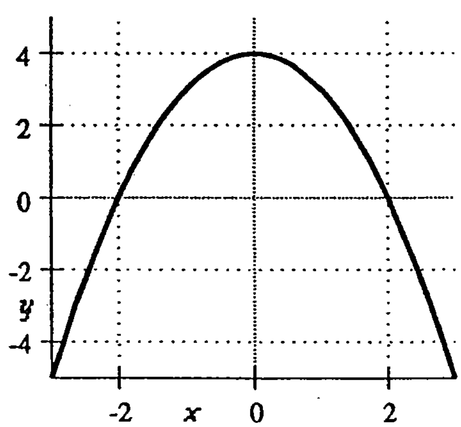
In other words, the graph of $|f(x)|$ lies entirely on and above the x-axis, $y \geq 0$ for all x.

To sketch the graph of $|f(x)|$:

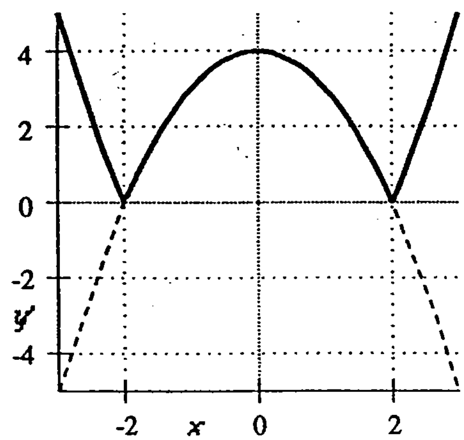
1. **Sketch the graph of $f(x)$.**
2. **Reflect across the x-axis any portion of the graph of f that lies below the x-axis.**

EXAMPLE 1.   Let $f(x) = 4 - x^2$. Sketch the graphs of $f(x)$ and $| f(x) |$.

$f(x) = 4 - x^2$ is a parabola that opens down with y-intercept 4 and x-intercepts ± 2.



$y = 4 - x^2$



$y = | 4 - x^2 |$

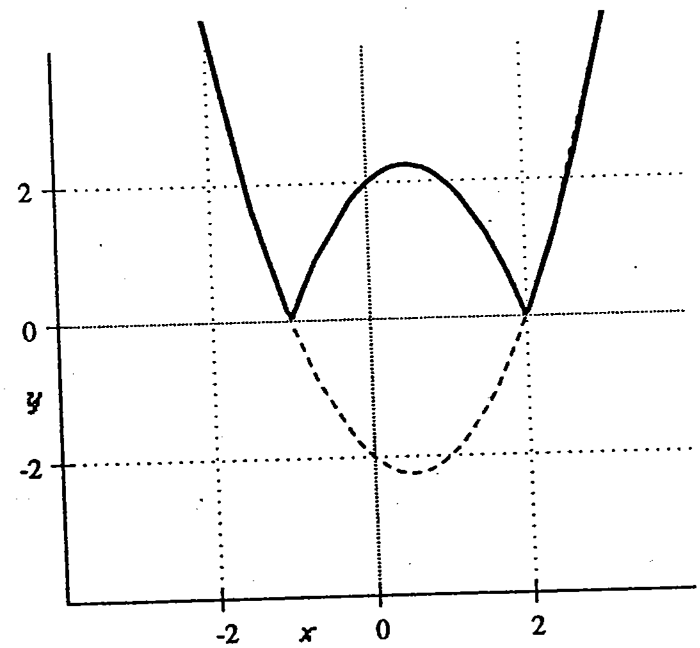
Graphing can be an aid to rewriting an absolute value function without using absolute value notation (rather than going through the formal definition and solving an inequality). From the graphs above, we see that for $-2 \le x < 2$, $| f(x) |$ is the same as $f(x)$, and for $x < -2$ or $x > 2$, $| f(x) | = - f(x)$. Therefore,

$$|4 - x^2| = \begin{cases} 4 - x^2, & \text{for } -2 \le x \le 2 \\ x^2 - 4, & \text{for } x < -2 \text{ or } x > 2 \end{cases}$$

EXAMPLE 2. On the same set of axes, sketch the graphs of

$$f(x) = x^2 - x - 2 \quad \text{and} \quad g(x) = |f(x)| = |x^2 - x - 2|$$

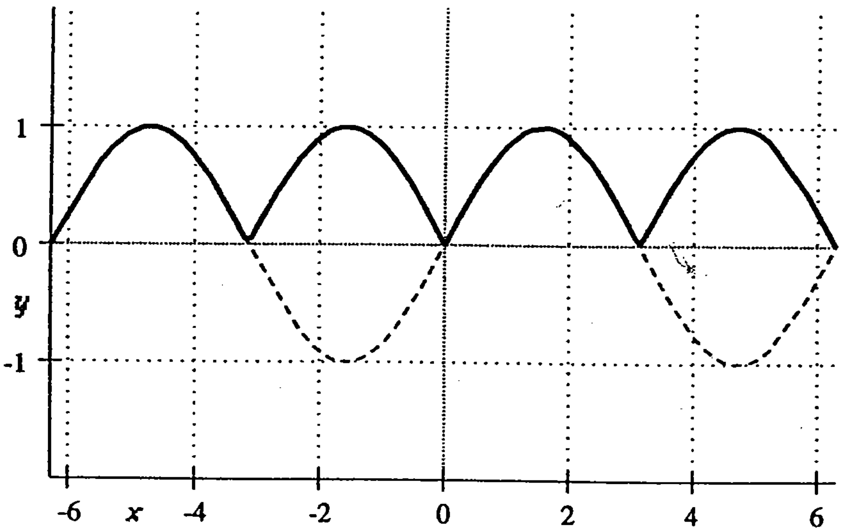
We first sketch $f(x) = x^2 - x - 2 = (x - 2)(x + 1)$. It is a parabola that opens up with y-intercept -2 and x-intercepts -1, 2. Then we reflect across the x-axis the portion of the graph of f that lies below the x-axis. The dashed line is the graph of f, the solid line is the graph of $|f|$.



Again, we use the graph as an aid in rewriting the absolute value function.
For $x \leq -1$ or $x \geq 2$, $|f(x)|$ is the same as $f(x)$.
For $-1 < x < 2$, $|f(x)|$ is the opposite of $f(x)$. Therefore

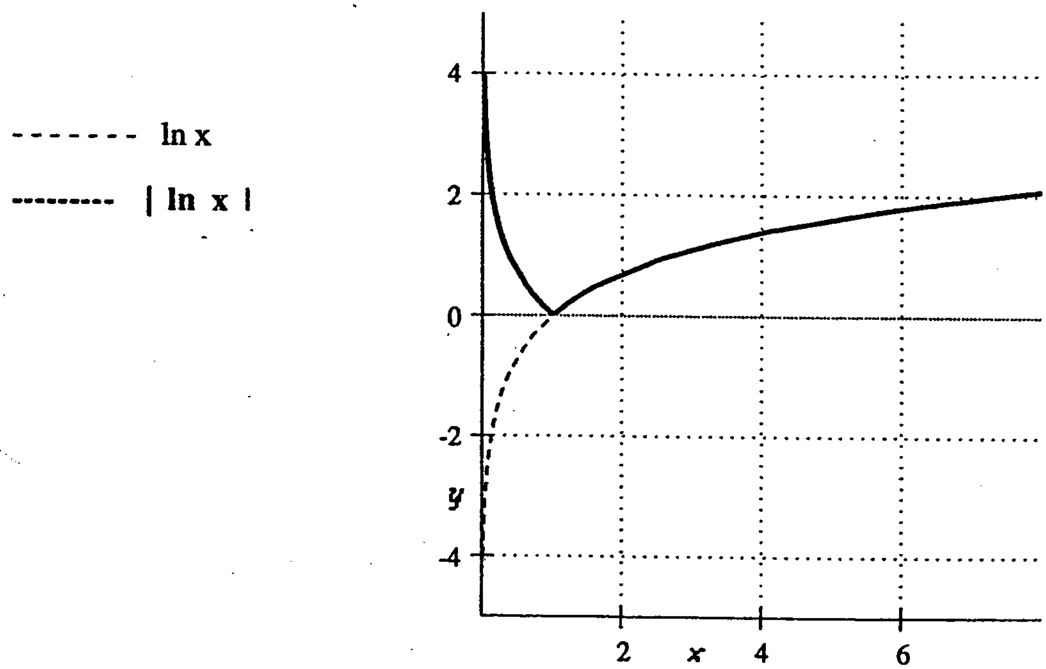
$$|x^2 - x - 2| = \begin{cases} x^2 - x - 2, & \text{for } x \leq -1 \text{ or } x \geq 2 \\ 2 + x - x^2, & \text{for } -1 < x < 2 \end{cases}$$

EXAMPLE 3. Sketch the graphs of sin x and | sin x | .



Note that although the period of sin x is 2π, the period of | sin x | is π.

EXAMPLE 4. Sketch the graphs of ln x and | ln x | .



THE GRAPH OF $f(|x|)$

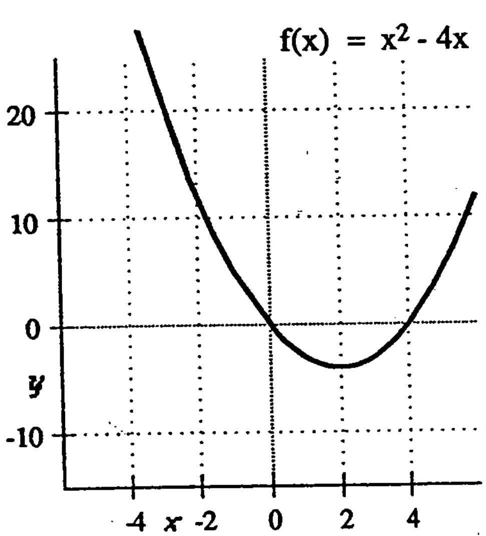
By the definition of absolute value,

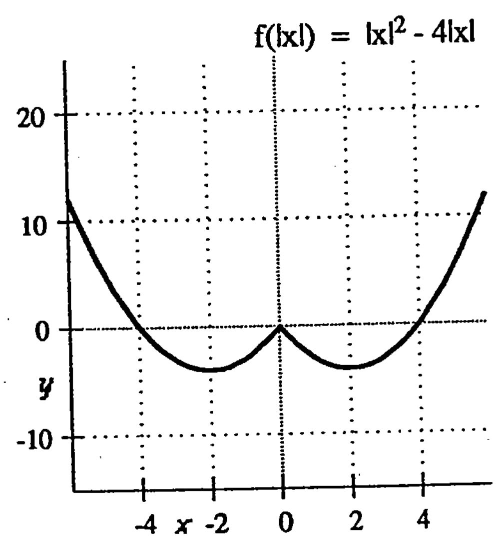
$$f(|x|) = \begin{cases} f(x), & \text{for } x \geq 0 \\ f(-x), & \text{for } x < 0 \end{cases}$$

From the above definition, we see that for positive or zero x, the graph of $f(|x|)$ is identical to that of $f(x)$. For negative x, the graph of $f(|x|) = f(-x)$, i.e., it is a reflection of $f(x)$ across the y-axis.

TO SKETCH THE GRAPH OF $f(|x|)$:

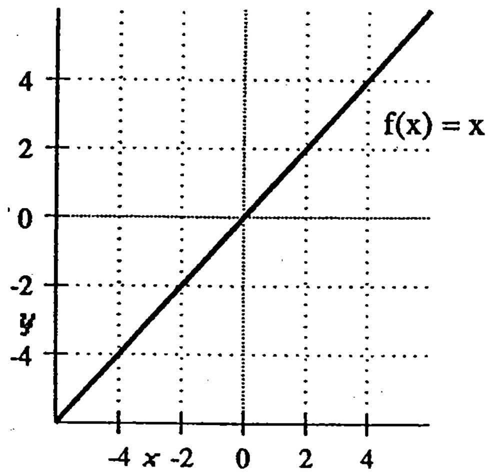
1. **Sketch the graph of $f(x)$ for $x \geq 0$.**
2. **Sketch the mirror image of the above portion of $f(x)$ across the y-axis.**

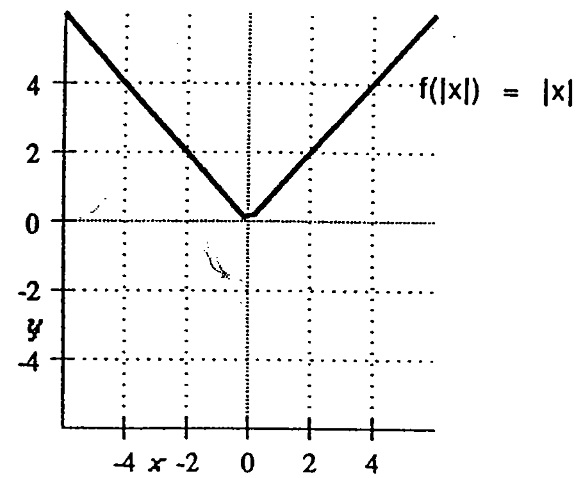




$$f(|x|) = |x|^2 - 4|x| = \begin{cases} x^2 - 4x, & \text{for } x \geq 0 \\ x^2 + 4x, & \text{for } x < 0 \end{cases}$$

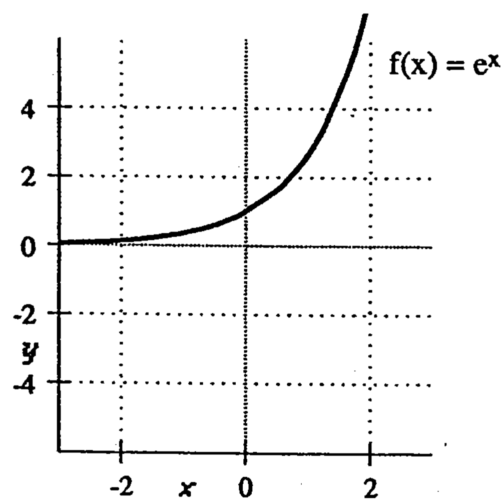
EXAMPLE. Sketch the graphs of $f(x) = x$ and $f(|x|) = |x|$

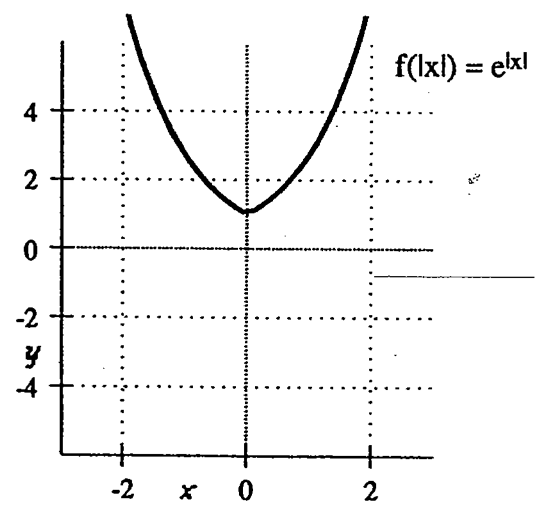




$$f(|x|) = |x| = \begin{cases} x, & \text{for } x \geq 0 \\ -x, & \text{for } x < 0 \end{cases}$$

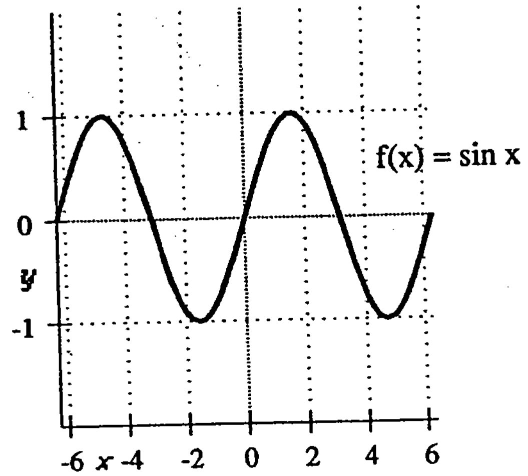
EXAMPLE. Sketch the graphs of $f(x) = e^x$ and $f(|x|) = e^{|x|}$

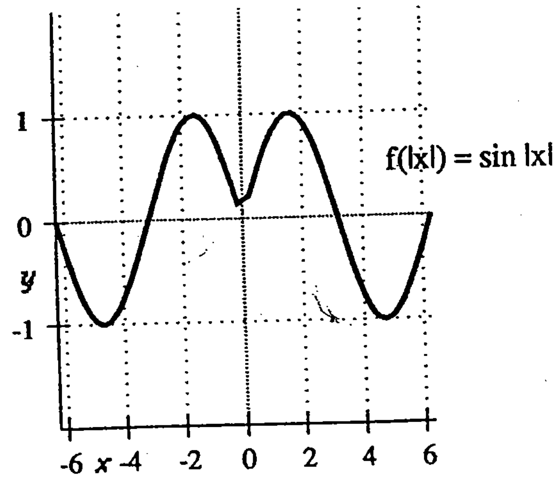




$$f(|x|) = e^{|x|} = \begin{cases} e^x, & \text{for } x \geq 0 \\ e^{-x}, & \text{for } x < 0 \end{cases}$$

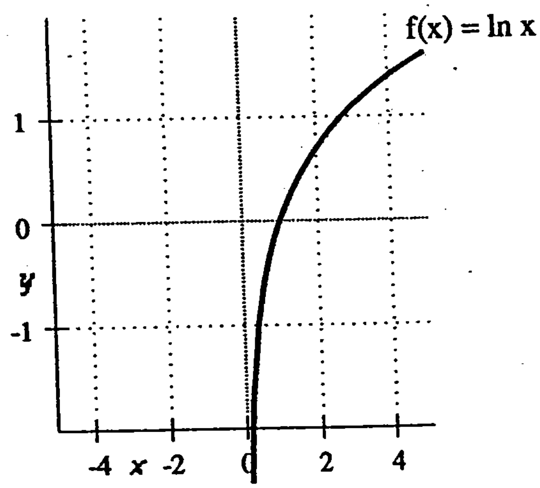
EXAMPLE. Sketch the graphs of $f(x) = \sin x$ and $f(|x|) = \sin |x|$

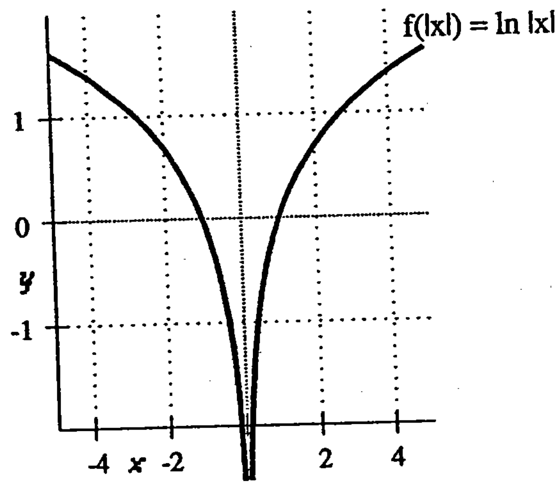




$$f(|x|) = \sin |x| = \begin{cases} \sin x, & \text{for } x \geq 0 \\ \sin -x = -\sin x, & \text{for } x < 0 \end{cases}$$

EXAMPLE. Sketch the graphs of $f(x) = \ln x$ and $f(|x|) = \ln |x|$





$$f(|x|) = \ln |x| = \begin{cases} \ln x, & \text{for } x \geq 0 \\ \ln -x, & \text{for } x < 0 \end{cases}$$

A COMPARISON OF THE GRAPHS OF $|f(x)|$ AND $f(|x|)$

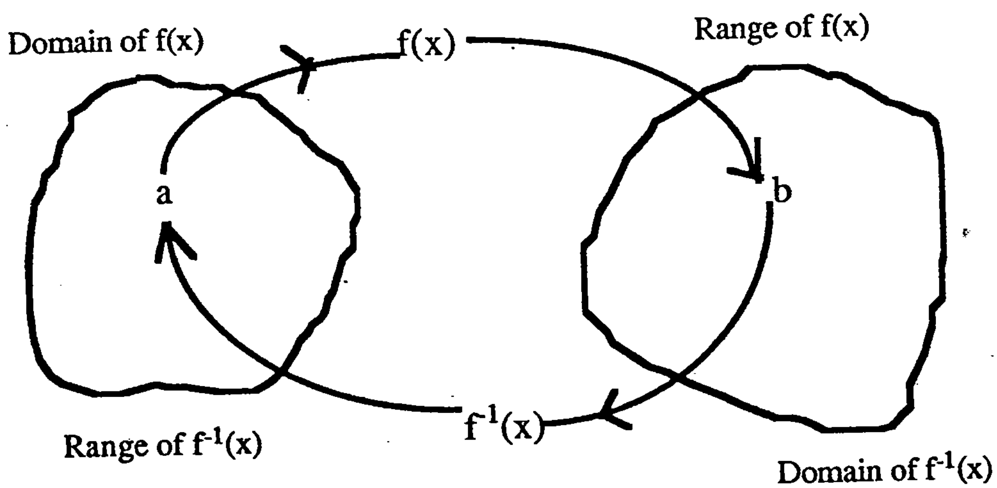
Unlike $|f(x)|$, which has *always* positive or zero y values, $f(|x|)$ can have negative y values, as you have seen from the graphs on previous pages.

$f(|x|)$ is *always* an even function and symmetric to the y-axis, because $f(|-x|) = f(|x|)$ for all x, whereas $|f(x)|$ may be odd, even, or neither.

INVERSE FUNCTIONS

Intuitively, the inverse of a function "undoes" what the function does. If the function squares a number, the inverse will take its square root. If a function doubles a number, the inverse will halve it. The notation for the inverse function of f(x) is **$f^{-1}(x)$**, (read as f inverse, not f to the -1). The -1 is a notation, not an exponent.

$$f^{-1}(x) \neq \frac{1}{f(x)}$$



The diagram above shows another way of looking at what the inverse of a function does. The function f acts on a number in its domain, a, and maps into b, which is in the range of f, i.e., f(a) = b. The function f^{-1} undoes this. It takes the number b in its domain (the domain of f^{-1} is the range of f) and maps it into a, $f^{-1}(b) = a$, which is in its range (the range of f^{-1} is the domain of f). Thus

$$f(f^{-1}(a)) = a \quad \textbf{and} \quad f^{-1}(f(a)) = a$$

If one function undoes what the other does, then if you start with a, you wind up with a.

FINDING THE INVERSE OF A FUNCTION

The fact that the domain and range of inverse functions are switched gives us a procedure for finding the inverse of a function. It is a two-step procedure.

1. **switch the x's and y's in the equation**
2. **solve for y**

EXAMPLE 1. Find the inverse of $f(x) = 3x - 1$

$$y = 3x - 1$$
$$x = 3y - 1$$
$$x + 1 = 3y$$
$$y = f^{-1}(x) = \frac{x + 1}{3}$$

EXAMPLE 2. Find the inverse of $f(x) = x^3 + 2$

$$y = x^3 + 2$$
$$x = y^3 + 2$$
$$x - 2 = y^3$$
$$y = f^{-1}(x) = \sqrt[3]{x - 2}$$

EXAMPLE 3. **Study this example carefully. Students always seem to have trouble remembering this technique.**

Find the inverse of $f(x) = \dfrac{x - 2}{2x + 1}$

$$y = \frac{x - 2}{2x + 1}$$

$$x = \frac{y - 2}{2y + 1}$$

$$x\,(2y + 1) = y - 2$$

Here is where students usually first get stuck on this kind of problem. Let's think about it. We wish to solve for y, that is, we wish to get y by itself on one side, and every non-y term on the other side. First we distribute x on the left side.

$$2xy + x = y - 2$$

Having survived the previous hurdle, here is the second place students usually get stuck. If we keep our objective in mind, it's easy. We want y by itself on one side. So get all the terms containing y on one side, and all the non-y terms on the other side.

$$2xy - y = - x - 2$$

And here is where 90% of the 10% who make it this far give up. What to do? Well, you want y by itself? So extricate it from this mess. **Factor it out!**

$$y\,(2x - 1) = -x - 2$$

The next step is obvious. Divide.

$$y = f^{-1}(x) = \frac{-x - 2}{2x - 1}$$

The next step is not necessary. It's for safety reasons only. Dangling negatives are dangerous. They often get lost in the shuffle; what students often refer to as "dumb mistakes." I don't know what they're talking about. I've been teaching for 20 years, and I've never seen a *smart* mistake. But I do know that he is called wise who minimizes the number of negative signs. To do this, multiply the right side by a form of 1 (one).

$$y = f^{-1}(x) = \frac{-x-2}{2x-1}\cdot\frac{-1}{-1} = \frac{x+2}{1-2x}$$

$$\text{Thus, } f^{-1}(x) = \frac{x+2}{1-2x}$$

Now go back to the previous page and go through the techniques for the solution again.

AND REMEMBER IT!

The Solution to a Vexing Problem: A Clever Way to Find the Range of a Rational Function

Generally, finding the domain of a function over the set of real numbers is relatively easy. We need to be on the look-out for division by zero, square roots of negative numbers, and definitions of some special functions. Finding the range of a function is sometimes not so easy. Often we need to resort, or think about, the graph of the function. We now have at our disposal another technique.

Since the range of f(x) is the domain of $f^{-1}(x)$, and domains are generally easier to find, we can find the range of f(x) by finding the domain of $f^{-1}(x)$.
How clever! See Example 4 that follows.

EXAMPLE 4. Find the domain and range of the function defined in Example 3 above:

$$f(x) = \frac{x - 2}{2x + 1}$$

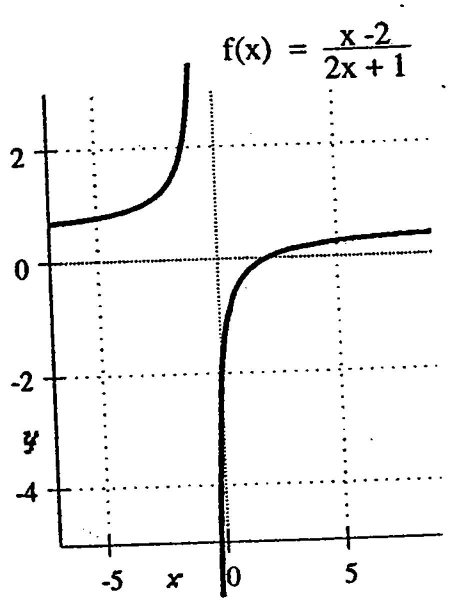
The domain of f is $x \neq -\frac{1}{2}$

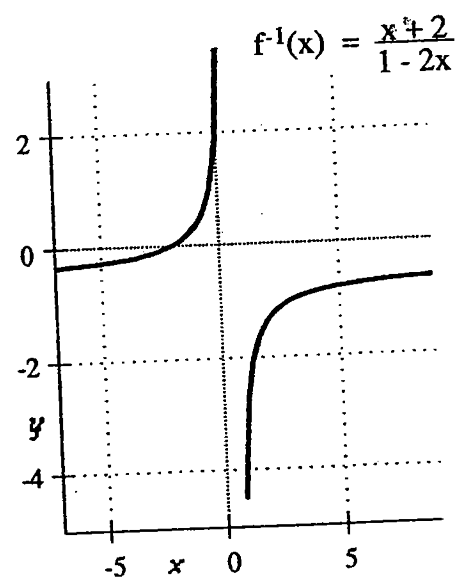
The range of f is the domain of $f^{-1}(x) = \frac{x + 2}{1 - 2x}$

$$y \neq +\frac{1}{2}$$

	Domain	Range
$f(x)$	$x \neq -\frac{1}{2}$	$y \neq +\frac{1}{2}$
$f^{-1}(x)$	$x \neq +\frac{1}{2}$	$y \neq -\frac{1}{2}$

The graphs of f and f^{-1} are shown below. Note the difference in the horizontal and vertical asymptotes.





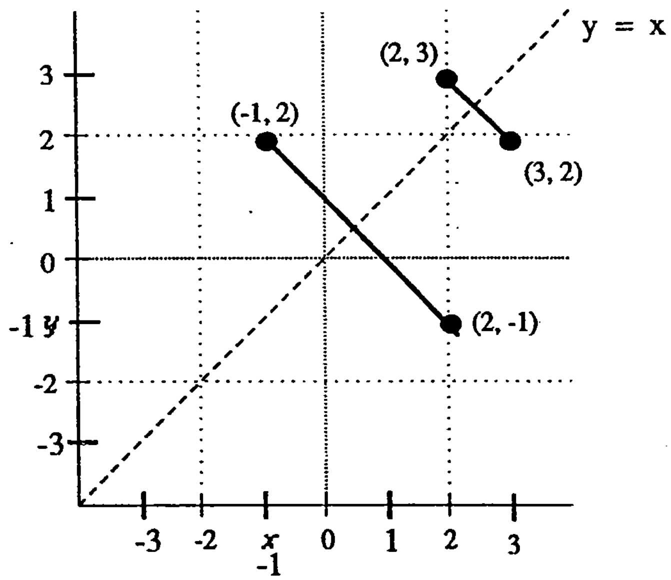
GRAPHS OF INVERSE FUNCTIONS

We have said, over and over again, that for inverse function, the domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1}. Again, this means that if $f(a) = b$, then $f^{-1}(b) = a$.

How does this concept relate the graphs of f and f^{-1}?

Let us take a look.

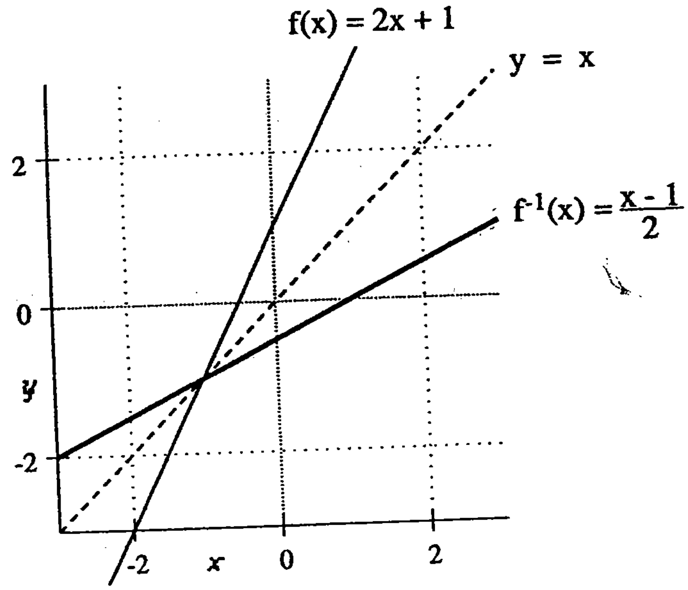
Suppose the point (2, 3) is on the graph of f. Then the point (3, 2) is on the graph of f^{-1}. If (-1, 2) is on the graph of f, then (2, -1) is on the graph of f^{-1}. Let us plot these points.

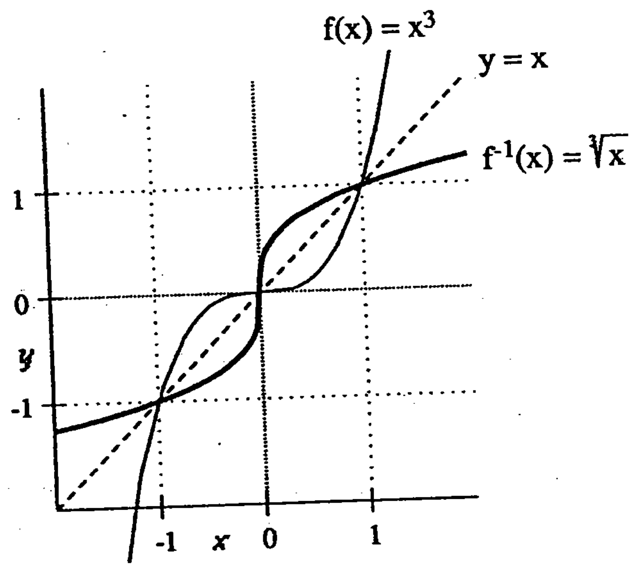


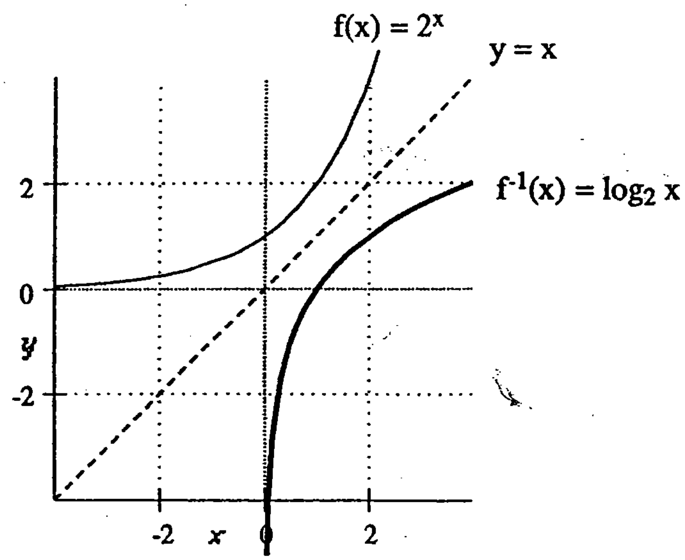
The dashed line in the graph above is the line $y = x$. If we consider $y = x$ as the surface of a mirror, then we see that corresponding points on f and f^{-1} seem to be mirror images of each other. This is true in general for all pairs (a, b) and (b, a). We can conclude the following:

The graphs of f and f^{-1} are mirror images of each other across the line $y = x$.

Below are the graphs of inverse pairs of functions



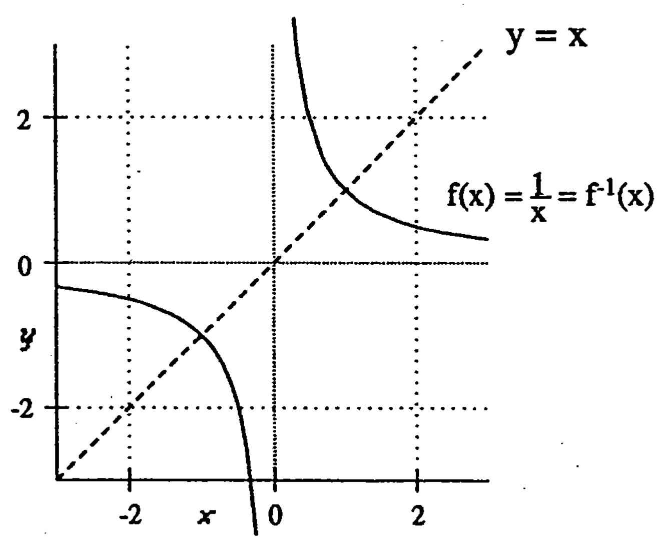




Some functions are their own inverses, such as

$$f(x) = \frac{1}{x} = f^{-1}(x)$$

Note that $f(f^{-1}(x)) = \dfrac{1}{\frac{1}{x}} = x$

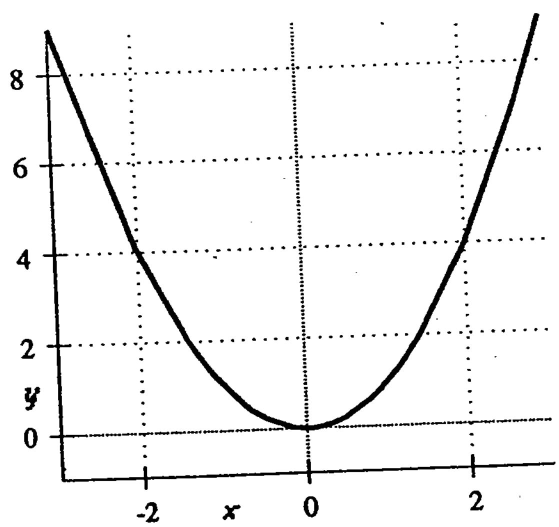


ONE-TO-ONE FUNCTIONS

DEFINITION OF A ONE-TO-ONE FUNCTION

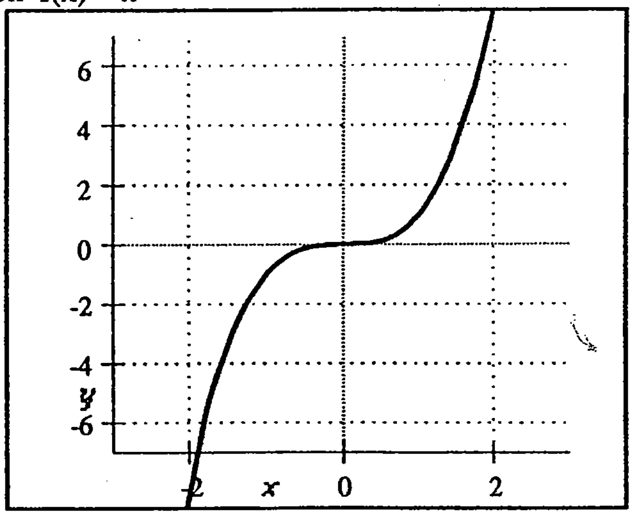
A function is said to be one-to-one if and only if for every value of x in its domain there is exactly one value of y in its range, and for every value of y in its range, there is exactly one value of x in its domain.

Consider $f(x) = x^2$, whose graph is shown below.



We see that although f is a function, and for every value of x there is only one value of y, it is not a one-to-one function because for each value of y there are two values of x. For instance, when $y = 4$, $x = \pm 2$. Note that although the graph of f passes the vertical line test for a function, it does not pass a horizontal line test, that is, a horizontal line intersects the graph at more than one point.

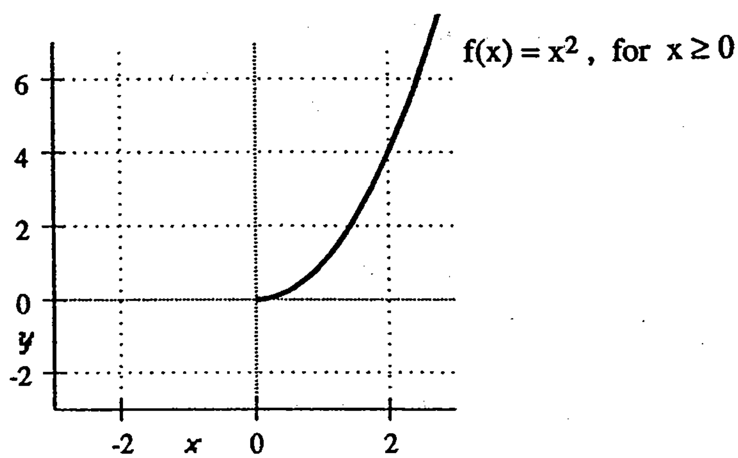
Now consider the function $f(x) = x^3$



The above function fits our definition of a one-to-one function given on the previous page. Also note that all vertical and horizontal lines do not intersect its graph at more than one point. This gives us an easy method for determining whether a function is one-to-one.

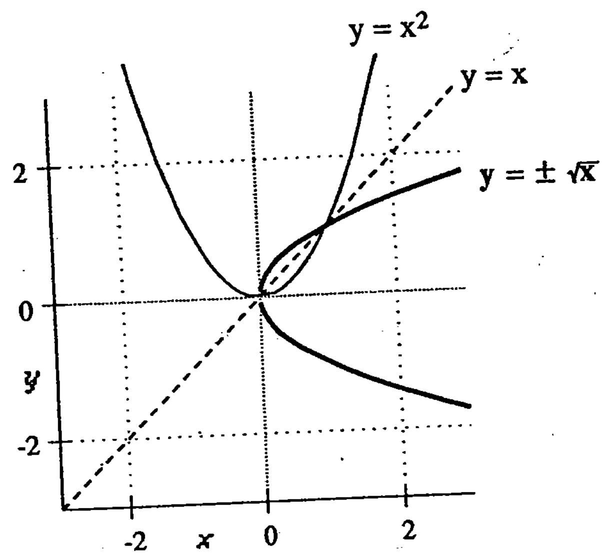
A function is one-to-one if it passes both the vertical and horizontal line test, i.e., if all vertical and horizontal lines do not intersect its graph at more than one point.

Returning to the function $f(x) = x^2$ on the previous page. WE could make it one-to-one by restricting its domain to $x \geq 0$. Its graph would then look like the one shown below. We will use this technique to ensure that the inverse of a function is also a function.



When is the inverse of a function also a function?

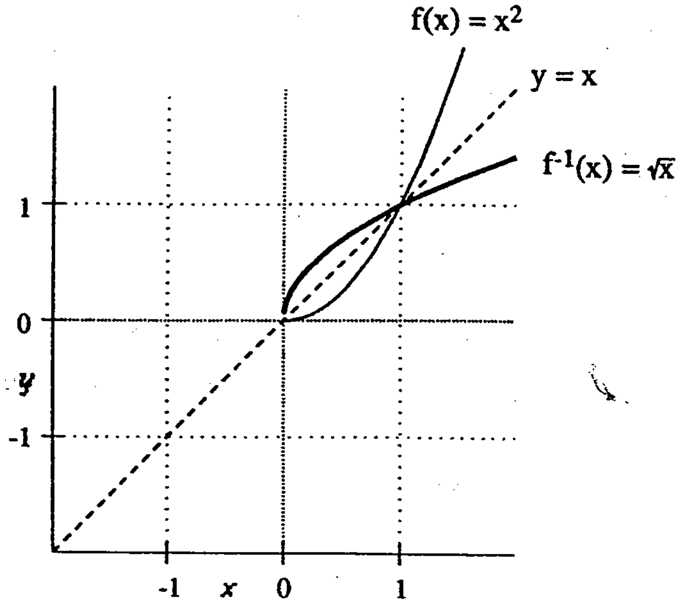
Let us investigate. Consider the graph of $f(x) = x^2$ below. We draw its mirror image across the line $y = x$.



We see that although $y = x^2$ is a function, its inverse, $y = \pm\sqrt{x}$ is not because it does not pass the vertical line test. The problem is that $y = x^2$ is not a one-to-one function. Whenever this is the case, the inverse will not be a function.

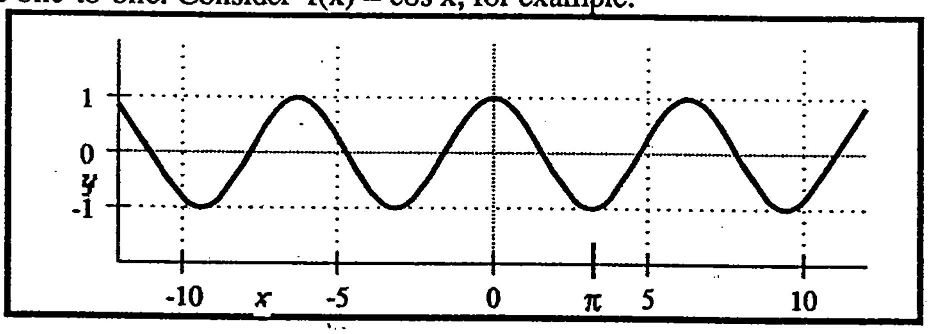
A FUNCTION MUST BE ONE-TO-ONE IN ORDER FOR ITS INVERSE TO BE A FUNCTION.

We can, however, restrict the domain of $f(x) = x^2$, thereby making it a one-to one function. If we define $f(x) = x^2$ for $x \geq 0$ only, then its graph is simply the right portion of the parabola, and it will pass the horizontal line test. Its inverse will then also be a function.

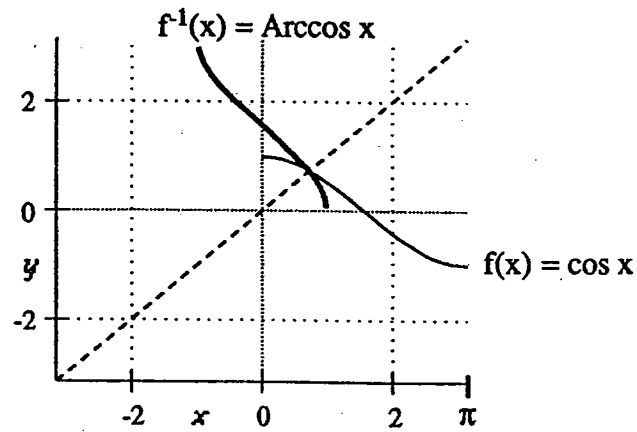


Thus, for $x \geq 0$, the inverse of $f(x) = x^2$, $f^{-1}(x) = \sqrt{x}$, is a function.

We will use this technique of restricting the domain to make a function one-to-one when we deal with the trigonometric functions. Because of their periodic nature, none of the trigonometric functions are one-to-one. Consider $f(x) = \cos x$, for example:

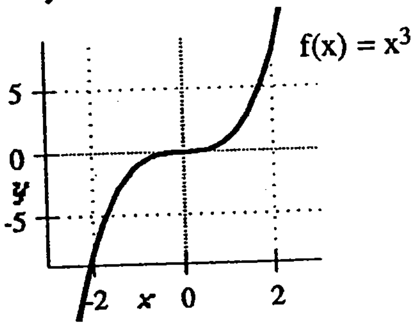


If we restrict the domain of cos x to $0 \leq x \leq \pi$, then it becomes one-to one, and its inverse, Arccos x, will be a function.

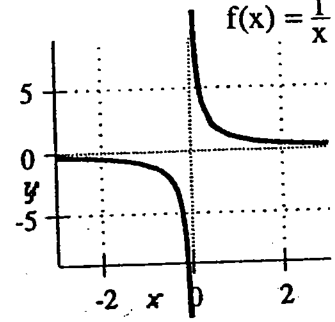


A FEW FINAL ODDS AND ENDS

If a function is everywhere increasing or everywhere decreasing over its domain, then it is one-to-one.

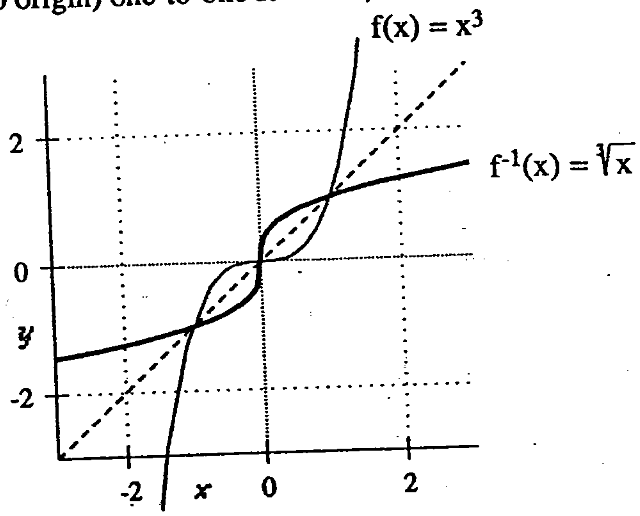


everywhere increasing over its domain

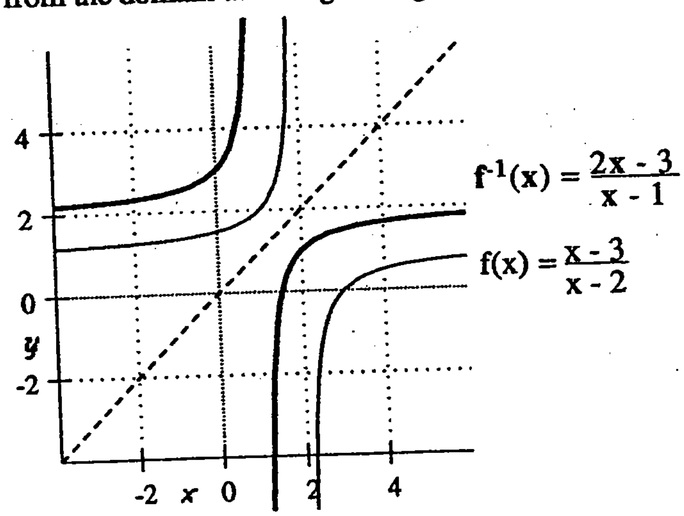


everywhere decreasing over its domain

If $f(x)$ is an odd (symmetric to origin) one-to-one function, then $f^{-1}(x)$ will also be an odd function.



If $f(x)$ is one-to-one, and its graph has asymptotes $x = a$ and $y = b$, then the graph of $f^{-1}(x)$ has asymptotes $x = b$ and $y = a$; if $f(x)$ has x-intercept c and y-intercept d, then $f^{-1}(x)$ has x-intercept d and y-intercept c (this follows from the domain and range being switched).



SUMMARY OF THE PROPERTIES OF INVERSE FUNCTIONS

Let $g(x)$ be the inverse of $f(x)$, that is, $g(x) = f^{-1}(x)$. Then

1. $f(g(x)) = g(f(x)) = x$

2. The graphs of f and g are mirror images of each across the line $y = x$

3. If $f(a) = b$, then $g(b) = a$
   The domain and range of f and g are switched. The domain of f is the range of g, and the range of f is the domain of g.

   From Property 3, it follows that:
   a. If $x = a$ and $y = b$ are the respective x and y-intercepts of the graph of f, then $x = b$ and and $y = a$ are the respective x and y-intercepts of the graph of g.
   b. If $x = c$ and $y = d$ are respectively the vertical and horizontal asymptotes of the graph of f, then $x = d$ and $y = c$ are the respective vertical and horizontal asymptotes of the graph of g.

4. If f is an odd function, then g is an odd function.

To find the inverse of a function
1. Switch the x's and y's
2. Solve for y

FATAL ERROR: $f^{-1}(x) \neq \frac{1}{f(x)}$